Vienna University of Technology SS 2019
Institute for Analysis and Scientific Computing
Dr. B. Nectoux Prof. A. Arnold

Exercise Sheet 7 "Nonlinear Partial Differential Equations"
(parabolic PDEs)

Exercise 1. Let $\Omega \subset \mathbb{R}$ be a bounded domain with smooth boundary. Show that, for $u_0 \in L^2(\Omega)$, the initial/boundary-value problem

$$\begin{cases} u_t - u_{xx} = \arctan(u)u_x & \text{in } \Omega \times (0,\infty)\,, \\ u = 0 & \text{on } \partial\Omega \times [0,\infty)\,, \\ u(\cdot, t=0) = u_0 & \text{in } \Omega\,, \end{cases}$$

has a unique global solution $u \in C([0,\infty), L^2(\Omega))$.

Hint: Use Theorems 3.15 and 3.13 from the lecture. Look for a function g with $g'(x) = \arctan(x)$.

Exercise 2. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and $T > 0$. Consider the Cauchy problem

$$\begin{cases} u_t - \Delta u = -u|u| & \text{in } \Omega \times (0,T]\,, \\ u = 0 & \text{for } (x,t) \in \partial\Omega \times [0,T]\,, \\ u(t=0,\cdot) = u_0 & \text{in } \Omega\,, \end{cases} \tag{1}$$

with $u_0 \in L^2(\Omega)$. Prove, for $n = 2$ and $n = 3$, that a unique weak solution $u \in C([0,T], L^2(\Omega))$ exists.

Attention: If $n > 1$ then $u|u| \in H^{-1}(\Omega)$ does not hold for all $u \in L^2(\Omega)$!
Hint: Use a fixed-point iteration $A(u) = w$ with $-|u(t)|w(t)$ as a right-hand-side and a suitable function subspace, and use the Gagliardo-Nirenberg*-inequality in your estimates: For $1 \le p, r, q \le \infty$ it holds*

$$||u||_{L^p(\Omega)} \le C||\nabla u||^{\alpha}_{L^r(\Omega)}||u||^{1-\alpha}_{L^q(\Omega)}, \quad \tfrac{1}{p} = \alpha\big(\tfrac{1}{r} - \tfrac{1}{n}\big) + (1-\alpha)\tfrac{1}{q}, \quad \alpha \in [0,1].$$

Finally you can use the techniques of the proof of Theorem 3.16.

Exercise 3. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Consider the Cauchy problem

$$\begin{cases} u_t - \Delta u = f(u) & \text{in } \Omega \times (0,T]\,, \\ \frac{\partial u}{\partial \nu} = 0 & \text{for } (x,t) \in \partial\Omega \times [0,T]\,, \\ u(t=0,\cdot) = u_0 & \text{in } \Omega\,, \end{cases} \tag{2}$$

where $u_0 \in L^\infty(\Omega)$, and f is a continuous, monotone decreasing function and there exists $M_0 > 0$ such that $f(M_0) \leq 0$ and $f(-M_0) \geq 0$. Suppose $V = H^1(\Omega)$, $H = L^2(\Omega)$ and $u \in L^2(0,T;V) \cap H^1(0,T;V')$ is a weak solution of (2). Show that:

$$\|u(t,\cdot)\|_{L^\infty(\Omega)} \leq \max\{M_0, \|u_0\|_{L^\infty(\Omega)}\}, \quad t > 0.$$

Hint: Use $v_1 := (u-M)^+$ and $v_2 := (u+M)^-$ for suitable M as test functions in the weak formulation.

Exercise 4. Let $\Omega \subset \mathbb{R}^n$ $(n \leq 3)$ be a bounded domain with smooth boundary, $\epsilon > 0$ and let u be a classical solution of the *Allen-Cahn-Equation*

$$\begin{cases} u_t - \epsilon\Delta u = -\frac{1}{\epsilon}(u^2-1)u & \text{in } \Omega,\ t>0, \\ \nabla u \cdot \nu = 0 & \text{auf } \partial\Omega,\ t > 0, \\ u(0) = u_0. \end{cases} \tag{3}$$

Define for $v \in H^1(\Omega)$ the functional

$$E[v] := \frac{\epsilon}{2}\int_\Omega |\nabla v|^2 dx + \frac{1}{4\epsilon}\int_\Omega (v^2-1)^2 dx.$$

Show that

(i) The derivative of $t \mapsto E(u(t))$ is nonpositive for every $t > 0$.

(ii) $E : H^1(\Omega) \to \mathbb{R}$ is weakly lower semicontinuous, i.e. $u_k \rightharpoonup u$ weakly in $H^1(\Omega)$ implies $E(u) \leq \liminf_{k\to\infty} E(u_k)$.

Solutions will be discussed on Monday 9th of May 2019.