Beispiel 1.1

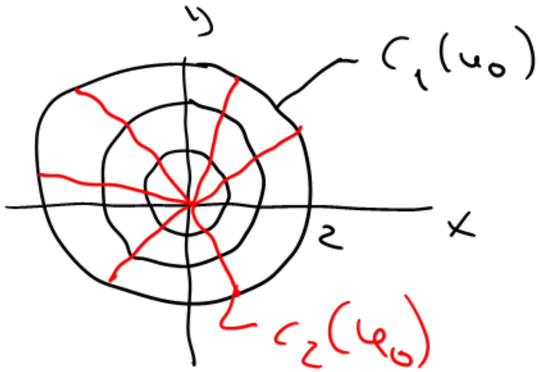
$$F = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u\cos\varphi \\ u\sin\varphi \\ \varphi \end{pmatrix},\ u \in [-2,2],\ \varphi \in [0,2\pi] \right\}$$

- Grundriss

$$\left\{ \begin{aligned} x &= u\cos\varphi \\ y &= u\sin\varphi \end{aligned} \right.,\ u \in [-2,2],\ \varphi \in [0,2\pi] \Big\}$$

- u_0 fest $\Rightarrow$ $\left. \begin{aligned} x &= u_0\cos\varphi \\ y &= u_0\sin\varphi \end{aligned} \right\} \Rightarrow x^2 + y^2 = u_0^2 = r^2$

$\Rightarrow r = |u_0|$

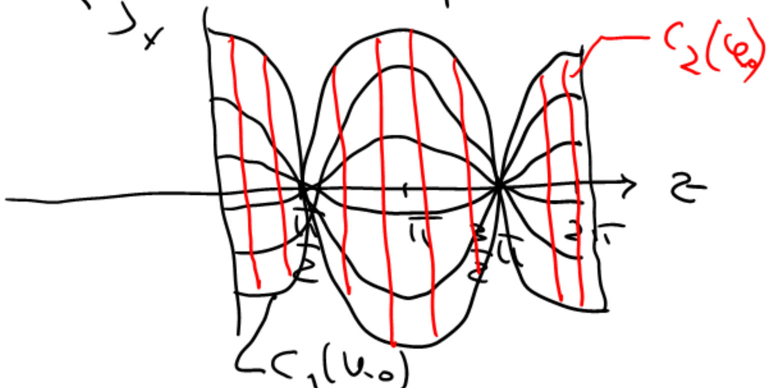


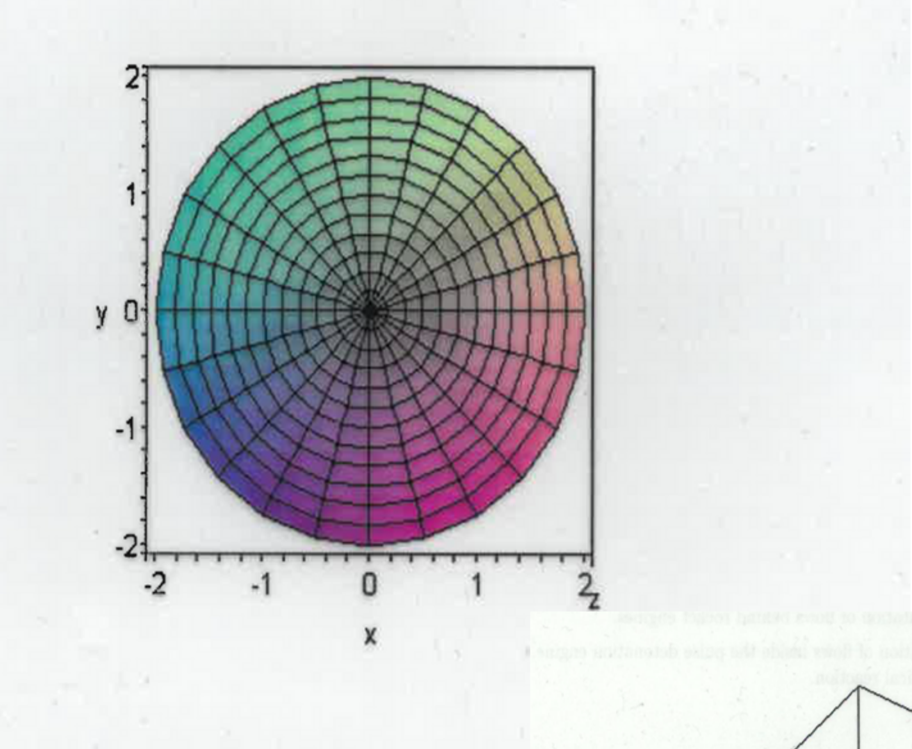
- φ_0 fest

- Aufriss: $x = u\cos\varphi,\ z = \varphi,\ u \in [-2,2],\ \varphi \in [0,2\pi]$

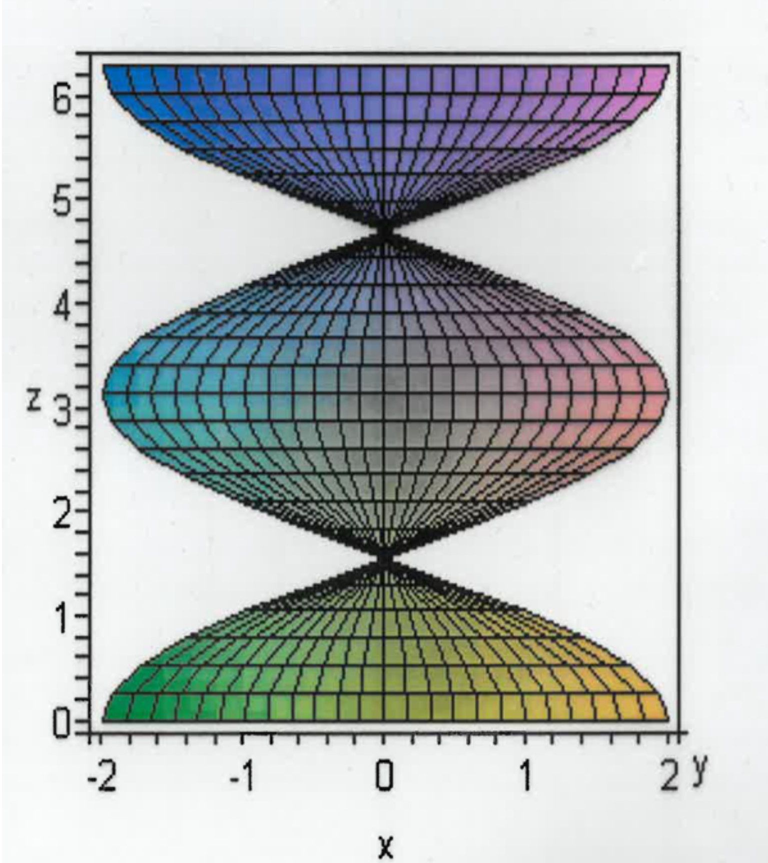
$x = u\cos z$

- u_0 fest
- z fest

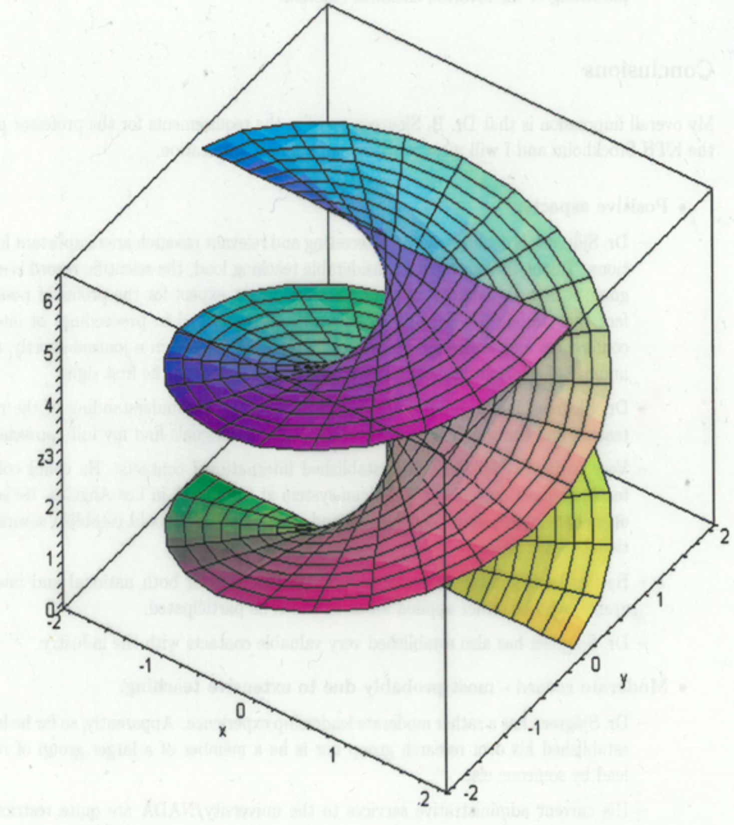




2
1
y 0
-1
-2
-2
-1
0
1
2
z
x



6
5
4
z 3
2
1
0
-2
-1
0
1
2
y
x



6
5
4
z3
2
1
0
-2
-1
0
x
1
2
2
1
0
y
-1
-2

Tangentialebene

$u = 1, \ \varphi = \frac{\pi}{6}$

$$F = \left\{ r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u\cos\varphi \\ u\sin\varphi \\ \varphi \end{pmatrix}, \ \varphi \in [0, 2\pi], \ u \in [-2, 2] \right\}$$

$$\begin{cases} \cos\frac{\pi}{6} = \cos 30^\circ = \frac{\sqrt{3}}{2} \\ \sin\frac{\pi}{6} = \sin 30^\circ = \frac{1}{2} \end{cases}$$

TE:

$$\left.\begin{aligned} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= r\left(1, \frac{\pi}{6}\right) + \frac{\partial r}{\partial u}\left(1, \frac{\pi}{6}\right)(u-1) + \frac{\partial r}{\partial \varphi}\left(1, \frac{\pi}{6}\right)\left(\varphi - \frac{\pi}{6}\right) \\ \frac{\partial r}{\partial u} &= \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}, \ \frac{\partial r}{\partial \varphi} = \begin{pmatrix} -u\sin\varphi \\ u\cos\varphi \\ 1 \end{pmatrix} \end{aligned}\right\} \Rightarrow$$

$$\boxed{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \\ \frac{\pi}{6} \end{pmatrix} + \underbrace{\alpha}_{u-1} \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} + \underbrace{\beta}_{\varphi - \frac{\pi}{6}} \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 1 \end{pmatrix} \quad \alpha, \beta \in \mathbb{R}}$$

Allgemein: $r(u,v)$ ... Fläche

TE: im Punkt (u_0,v_0)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = r(u_0,v_0) + (u-u_0)\frac{\partial r}{\partial u}(u_0,v_0) + (v-v_0)\frac{\partial r}{\partial v}(u_0,v_0)$$

Beispiel 1.2

$$F = \{ r(u,\varphi) = (u\cos\varphi, u\sin\varphi, \varphi), u \in [-2,2], \varphi \in [0,2\pi] \}$$

$$\text{Flächeninhalt} = \int_F dS = \int_B \underbrace{\|n(u,v)\| d(u,v)}_{},$$

wobei $\|n(u,v)\| = \sqrt{\operatorname{Det} M(u,v)}$; $\frac{\partial r}{\partial u} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}$, $\frac{\partial r}{\partial \varphi} = \begin{pmatrix} -u\sin\varphi \\ u\cos\varphi \\ 1 \end{pmatrix}$

$$\Downarrow$$

$$M(u,v) = \begin{pmatrix} \frac{\partial r}{\partial u}\cdot\frac{\partial r}{\partial u} & \frac{\partial r}{\partial u}\cdot\frac{\partial r}{\partial \varphi} \\ \frac{\partial r}{\partial u}\cdot\frac{\partial r}{\partial \varphi} & \frac{\partial r}{\partial \varphi}\cdot\frac{\partial r}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} 1 & \emptyset \\ \emptyset & u^2+1 \end{pmatrix}$$

$$\Rightarrow \quad \sqrt{\det M(u,v)} = \sqrt{u^2+1}$$

$$\Rightarrow FI = \int_{-2}^{2}\int_{0}^{2\pi} \sqrt{1+u^2}\, du\, d\varphi = 2\pi \int_{-2}^{2} \sqrt{1+u^2}\, du =$$

$$= \cancel{2}\pi \cdot \frac{1}{\cancel{2}} \left(u\sqrt{1+u^2} + \ln\left(u+\sqrt{1+u^2}\right)\right)\Big|_{-2}^{2} =$$

$$= \pi \left(2\sqrt{5} + \ln\left(2+\sqrt{5}\right) - \left(-2\sqrt{5} + \ln\left(-2+\sqrt{5}\right)\right)\right) \approx \underline{\underline{37.17}}$$

Beispiel 1.3 $F = \left\{ r(u,v) = \begin{pmatrix} u+v \\ u-v \\ 4uv \end{pmatrix}, (u,v) \in B \right\}$

Kurve in B: $C_B = \{ \underbrace{u(t) = v(t) = t}, t \in [0,1] \}$ $\Rightarrow$

Flächenkurve

$$C = \left\{ r(u(t), v(t)) = r(t) = \begin{pmatrix} 2t \\ 0 \\ 4t^2 \end{pmatrix}, t \in [0,1] \right\}$$

1. Methode $\quad r'(t) = \begin{pmatrix} 2 \\ 0 \\ 8t \end{pmatrix} \Rightarrow \|r'(t)\| = \sqrt{4 + 64t^2}$

$$\Rightarrow L = \int_0^1 \sqrt{4+64t^2}\,dt = 2\int_0^1 \sqrt{1+16t^2}\,dt =$$

$$\left\{ \begin{aligned} \sigma &:= 4t \\ d\sigma &= 4\,dt \Rightarrow dt = \tfrac{1}{4}\,d\sigma \end{aligned} \right.$$

$$= \cancel{2} \cdot \frac{1}{\cancel{4}\,2} \int \sqrt{1+\sigma^2}\,d\sigma = \frac{1}{2}\left(\frac{1}{2}\left(\sigma\sqrt{1+\sigma^2} + \ln\left(\sigma + \sqrt{1+\sigma^2}\right)\right)\right)$$

$$= \frac{1}{4}\left(4t\sqrt{1+16t^2} + \ln\left(4t+\sqrt{1+16t^2}\right)\right)\Big|_0^1 =$$

$$= \frac{1}{4}\left(4\sqrt{17} + \ln(4+\sqrt{17}\right) \approx \underline{4.65}$$

<u>2. Methode</u>

$$r(u,v) = \begin{pmatrix} u+v \\ u-v \\ 4uv \end{pmatrix} \Rightarrow$$

$$\frac{\partial r}{\partial u} = \begin{pmatrix} 1 \\ 1 \\ 4v \end{pmatrix}, \quad \frac{\partial r}{\partial v} = \begin{pmatrix} 1 \\ -1 \\ 4u \end{pmatrix} \Longrightarrow$$

$$M(u,v) = \begin{pmatrix} 2+16v^2 & 16uv \\ 16uv & 2+16u^2 \end{pmatrix}, \quad w(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} t \\ t \end{pmatrix} \Rightarrow w'(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

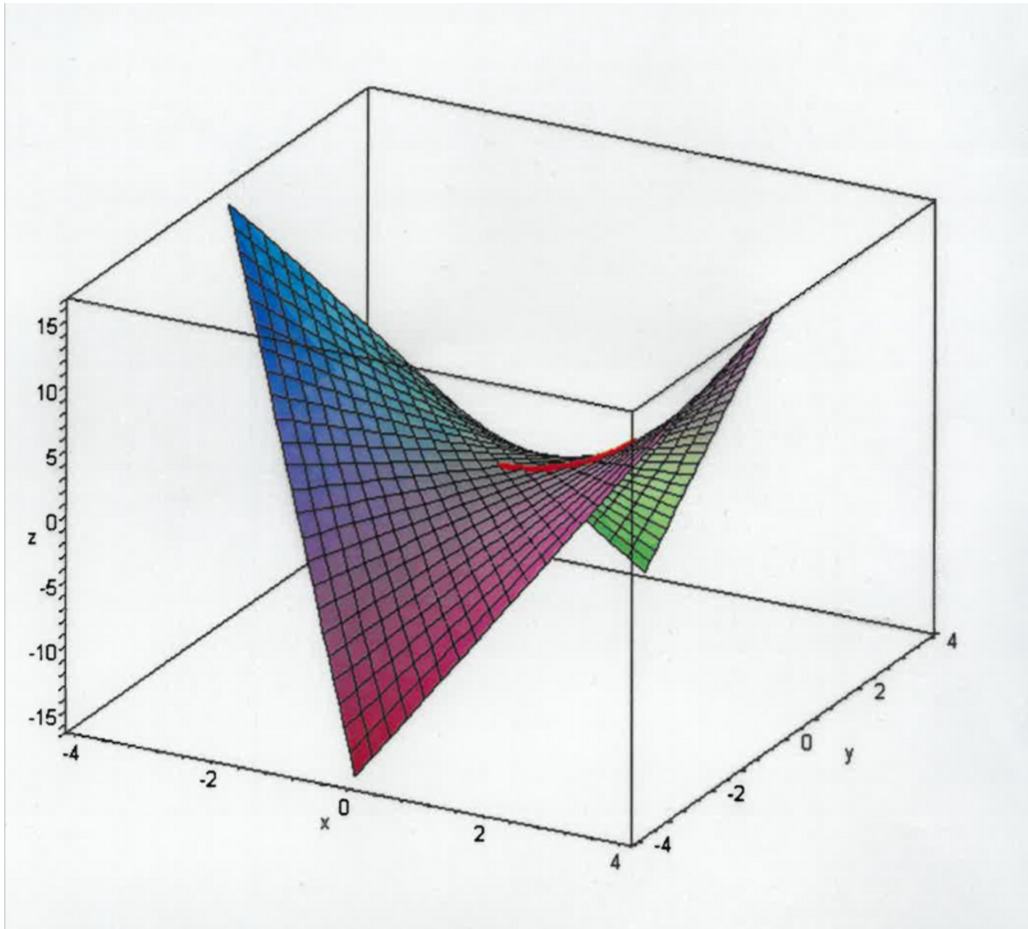
$$\Rightarrow \quad w'^T M(u,v) w' = (1, 1)\begin{pmatrix} 2+16v^2 & 16uv \\ 16uv & 2+16u^2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} =$$

$$= (1,1)\begin{pmatrix} 2+16v^2+16uv \\ 2+16u^2+16uv \end{pmatrix} =$$

$$= 4+16u^2+16v^2+32uv \Big|_{u=v=t} =$$

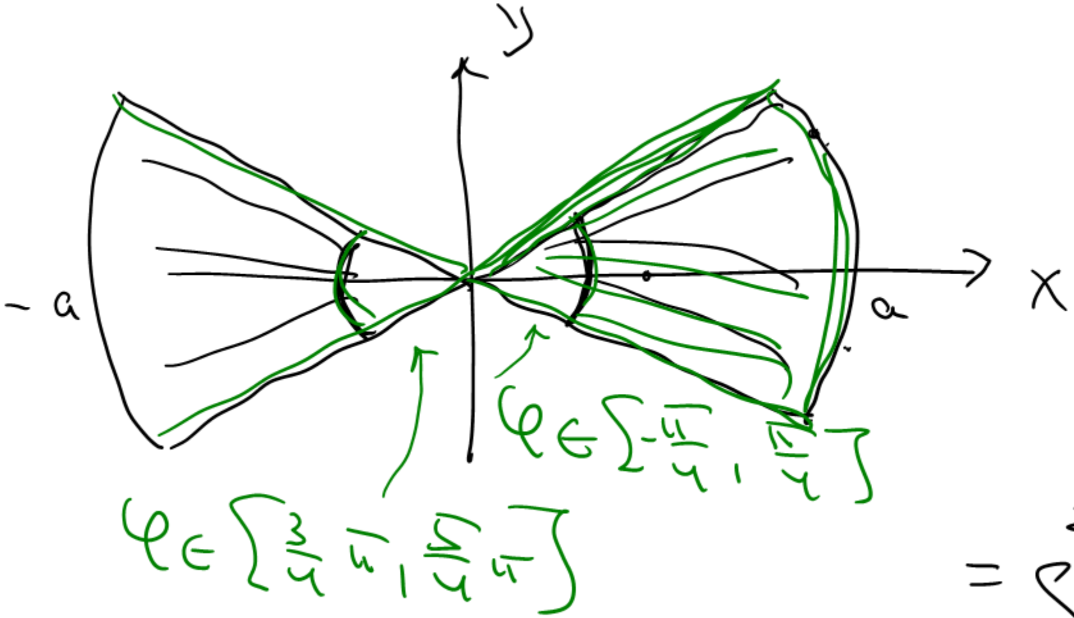
$$= 4+16t^2+16t^2+32t^2 = 4+64t^2$$

$$\Rightarrow \ell = \int_0^1 \sqrt{4+64t^2}\,dt$$



Beispiel 1.4

$$F = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \underline{y^2 + z^2 = x^2}, \; \underline{\underline{x^2 + y^2 \leq a^2}} \right\}$$



$$\begin{cases} x = \rho \cos\varphi \\ y = \rho \sin\varphi \end{cases}, \; \rho \in [0, a]$$

$$\Downarrow$$

$$z^2 = x^2 - y^2 = \rho^2 \cos^2\varphi - \rho^2 \sin^2\varphi$$

$$= \rho^2 (1 - \sin^2\varphi - \sin^2\varphi) =$$

$$= \rho^2 (1 - 2\sin^2\varphi) \Rightarrow$$

$$z = \pm \rho \sqrt{1 - 2\sin^2\varphi}$$

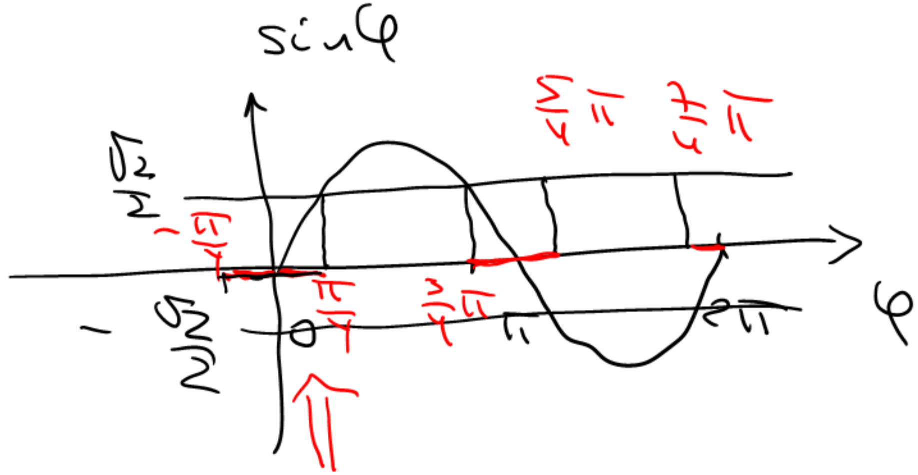
Betrachte $z \geq 0 \Rightarrow z = \rho \sqrt{1 - 2\sin^2\varphi}$

Einschränkung für φ:

$$1-2\sin^2\varphi \geq 0 \Rightarrow \sin^2\varphi \leq \frac{1}{2} \Rightarrow$$

$$\Rightarrow |\sin\varphi| \leq \frac{\sqrt{2}}{2}$$

$$\sin\varphi = \frac{\sqrt{2}}{2} \Rightarrow \varphi = \frac{\pi}{4}$$



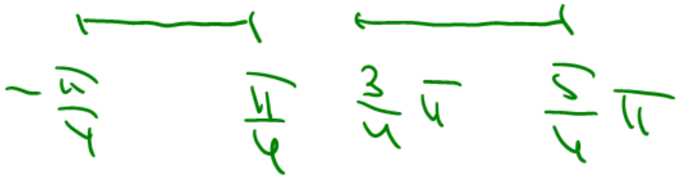
$z \geq 0$

$$r(\varrho,\varphi) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \varrho\cos\varphi \\ \varrho\sin\varphi \\ \varrho\sqrt{1-2\sin^2\varphi} \end{pmatrix}, \right.$$

$$\left. \varrho \in [0,a],\ \varphi \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \right\}$$

PD für $\frac{1}{4}$ der Fläche mit

$x \geq 0$ und $z \geq 0$.



$$\Rightarrow F_{\frac{1}{4}} = \int_{\varphi=-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{\varrho=0}^{a} \sqrt{\det M(\varrho,\varphi)}\, d\varrho\, d\varphi$$

$\cdot \quad \dfrac{\partial r}{\partial \rho} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ \sqrt{1-2\sin^2\varphi} \end{pmatrix}, \quad \dfrac{\partial r}{\partial \varphi} = \begin{pmatrix} -\rho\sin\varphi \\ \rho\cos\varphi \\ * \end{pmatrix}$

$$* = \rho \frac{1}{\cancel{2}\sqrt{1-2\sin^2\varphi}}(-\cancel{2}\,2\sin\varphi\cos\varphi)$$

$$= -\frac{\rho\sin 2\varphi}{\sqrt{1-2\sin^2\varphi}}$$

$$\Rightarrow M(\rho,\varphi) = \begin{pmatrix} 2\cos^2\varphi & -\rho\sin 2\varphi \\ -\rho\sin 2\varphi & \rho^2\left(1+\dfrac{\sin^2 2\varphi}{1-2\sin^2\varphi}\right) \end{pmatrix} \Rightarrow$$

$$\operatorname{Det} M(\rho,\varphi) = \frac{\rho^2\, 2\cos^2\varphi}{1-2\sin^2\varphi} \Rightarrow \sqrt{\operatorname{Det} M(\rho,\varphi)} = \frac{\sqrt{2}\,\rho|\cos\varphi|}{\sqrt{1-2\sin^2\varphi}}.$$

Da $\cos\varphi > 0$ für $\varphi \in [-\frac{\pi}{4}, \frac{\pi}{4}]$ folgt

$$\sqrt{\operatorname{Det} M(\rho,\varphi)} = \frac{\sqrt{2}\,\rho\cos\varphi}{\sqrt{1-2\sin^2\varphi}} \Rightarrow$$

$$\Rightarrow F_{1/4} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^a \sqrt{2}\,\rho \frac{\cos\varphi}{\sqrt{1-2\sin^2\varphi}}\, d\varphi\, d\rho =$$

$$= \sqrt{2}\, \frac{\rho^2}{2}\Big|_0^a \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos\varphi}{\sqrt{1-2\sin^2\varphi}}\, d\varphi =$$

$$= \frac{\sqrt{2}}{2} a^2 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos\varphi}{\sqrt{1-2\sin^2\varphi}}\, d\varphi$$

$u := \sqrt{2}\sin\varphi$

$du = \sqrt{2}\cos\varphi\, d\varphi$

$\Downarrow$

$$\cos\varphi\, d\varphi = \frac{du}{\sqrt{2}}$$

$$= \frac{1}{\cancel{\sqrt{2}}} \frac{\cancel{\sqrt{2}}}{2} a^2 \int_{u_1}^{u_2} \frac{du}{\sqrt{1-u^2}} =$$

$$= \frac{1}{2} a^2 \arcsin u \Big|_{u_1}^{u_2} = \frac{1}{2} a^2 \left(\arcsin\left(\sqrt{2}\sin(\varphi)\right) \right) \Big|_{-\frac{\pi}{4}}^{\pi/4}$$

$$= \frac{1}{2} a^2 \left(\arcsin\Big(\underbrace{\sqrt{2}\,\frac{\sqrt{2}}{2}}_{=1}\Big) - \arcsin\Big(\underbrace{\sqrt{2}\left(-\frac{\sqrt{2}}{2}\right)}_{=-1}\Big) \right)$$

$$= \frac{1}{2} a^2 \left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right) \right) = \frac{1}{2} a^2 \pi$$

$$\Rightarrow F = F_{\frac{1}{4}} \cdot 4 = \cancel{4} \cdot \frac{1}{\cancel{2}} a^2 \pi = \underline{\underline{2 a^2 \pi}}$$

<u>Alternative</u> $F = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \; y^2 + z^2 = x^2, \; x^2 + y^2 \le a^2 \right\}$

$$\left. \begin{aligned} y &:= \rho \cos\varphi \\ z &= \rho \sin\varphi \end{aligned} \right\} \Rightarrow x^2 = \rho^2 \cos^2\varphi + \rho^2 \sin^2\varphi = \rho^2 \Rightarrow x = \rho$$

$$\Rightarrow \left\{ r(\rho,\varphi) = \begin{pmatrix} \rho \\ \rho\cos\varphi \\ \rho\sin\varphi \end{pmatrix}, \varphi \in [0, 2\pi], \rho \in \left[0, \frac{a}{\sqrt{1+\cos^2\varphi}}\right] \right\}$$

(Das ist die $\frac{1}{2}$ Fläche $(x \geq 0)$.) Der Bereich für ρ:

$$x^2 + y^2 \leq a^2 \Rightarrow \rho^2 + \rho^2\cos^2\varphi \leq a^2 \Rightarrow$$

$$\rho^2(1+\cos^2\varphi) \leq a^2 \Rightarrow 0 \leq \rho^2 \leq \frac{a^2}{1+\cos^2\varphi} \Rightarrow$$

$$\boxed{0 \leq \rho \leq \frac{a}{\sqrt{1+\cos^2\varphi}}}$$

Weiter ist $\frac{\partial r}{\partial \rho} = \begin{pmatrix} 1 \\ \cos\varphi \\ \sin\varphi \end{pmatrix}, \frac{\partial r}{\partial \varphi} = \begin{pmatrix} 0 \\ -\rho\sin\varphi \\ \rho\cos\varphi \end{pmatrix} \Rightarrow$

$$M(\rho,\varphi) = \begin{pmatrix} 2 & 0 \\ 0 & \rho^2 \end{pmatrix} \Rightarrow \|m(\rho,\varphi)\| = \sqrt{2}\rho$$

Damit ist

$$F_{\frac{1}{2}} = \int_0^{2\pi} \int_0^{\frac{a}{\sqrt{1+\cos^2\varphi}}} \sqrt{2}\,\rho \, d\rho \, d\varphi$$ zu berechnen.

Nachtrag Bsp. 1.6.

z.z. $\int \sqrt{1+u^2}\,du = \frac{1}{2}\left(u\sqrt{1+u^2} + \ln\left(u+\sqrt{1+u^2}\right)\right)+c$

Beweis:

Es gilt $\cosh^2 t - \sinh^2 t = 1 \Rightarrow \cosh^2 t = 1+\sinh^2 t$

Subst. $\begin{cases} u := \sinh t \Rightarrow \\ du = \cosh t\,dt, \end{cases}$ $\sqrt{1+u^2} = \sqrt{1+\sinh^2 t} =$

$= \sqrt{\cosh^2 t} = |\cosh t| = \cosh t$

$$\Rightarrow \int \sqrt{1+u^2}\,du = \int \cosh t \cosh t\,dt = \underbrace{\int \cosh^2 t\,dt}_{I:=}$$

$$PI: \left\{ \begin{array}{l} g = \cosh t \Rightarrow dg = \sinh t\,dt \\ df = \cosh t\,dt \Rightarrow f = \sinh t \end{array} \right\} \Rightarrow$$

$$I = \cosh t \sinh t - \int \underbrace{\sinh}_{(\cosh^2 t - 1)}{}^2 t\,dt + C$$

$$\cosh t \sinh t + t - \underbrace{\int \cosh^2 t\,dt}_{= I} + C$$

$$2I = \cosh t \sinh t + t + C \Rightarrow$$

$$I = \int \sqrt{1+u^2}\,du = \int \cosh^2 t\,dt = \frac{1}{2}(\sinh t \cosh t + t) + C \Leftrightarrow$$

$$\boxed{I = \int \sqrt{1+u^2}\,du = \frac{1}{2}\left(u\sqrt{1+u^2} + t\right) + C}$$

Es gilt $\ln\left(u+\sqrt{1+u^2}\right) = \ln\left(\sinh t + \cosh t\right) =$

$$= \ln\left(\frac{e^t - e^{-t}}{2} + \frac{e^t + e^{-t}}{2}\right) = \ln e^t = t.$$

$\Rightarrow$

$$\Rightarrow \boxed{I = \int \sqrt{1+u^2}\,du = \frac{1}{2}\left(u\sqrt{1+u^2} + \ln\left(u+\sqrt{1+u^2}\right)\right) + C}$$