**7. Übung Höhere Wahrscheinlichkeitstheorie**

1. Hirsch's law: Let $f \geq 0$ be nondecreasing and $f(x)/x \downarrow 0$. Then

   $$\mathbb{P}(\max_{s \leq t} \beta(s) \geq f(t) \text{ infinitely often}) = 1$$

   if $\int t^{-3/2} f(t) dt = \infty$ and 0 otherwise.

   Lower bound: consider the events

   $$A_n = [\max_{s \leq 2^n} \beta(s) \leq f(2^{n+1}).$$

2. Continued: upper bound: Let

   $$T_1 = \inf\{t \geq 0 : \beta(t) \geq f(2)\},$$

   $$T_n = \inf\{t \geq T_{n-1} : \beta(T_{n-1} + t) - \beta(T_{n-1}) \geq f(2^n) - f(2^{n-1})\},$$

   and consider the events $[T_n \geq 2^n]$.

3. Show that

   $$\limsup_{t \to 0} \frac{\beta(t)}{\sqrt{2t \log\log(1/t)}} = 1.$$

4. Let

   $$\eta_T(t) = \frac{1}{\sqrt{2T \log\log T}} (\beta(tT) - t\beta(T)).$$

   Show that the set of accumulation points of this family is

   $$\{f \in \mathcal{S} : f(1) = 0\}$$

   gegeben ist.

5. Calculate

   $$\limsup \frac{\beta(t) + \beta(t/2)}{\sqrt{2t \log\log t}}.$$

6. Calculate

   $$\limsup \frac{\int_0^t s\beta(s) ds}{\sqrt{2t^3 \log\log t}}.$$

7. Calculate

   $$\limsup \int_0^1 \eta_t(s) ds.$$

8. Calculate

   $$\limsup \int_0^1 \eta_t(s)^2 ds.$$