Exercise 9

Given a probability semi-group $(P_t)_{t\geq 0}$ (family of continuous linear operators on $C(S)$ (by this I mean continuous functions from S to $\mathbb{R}$ vanishing at infinity) satisfying i) $P_0 f = f$, ii) $\lim_{t\to 0} P_t f = f$, iii) $P_s P_t f = P_{s+t} f$, iv) $P_t f \geq 0$ if $f \geq 0$, and v) $P_t 1 = 1$ (S compact case)), we defined the associated resolvent as

$$U_\alpha f := \int_0^\infty e^{-\alpha t} P_t f \mathrm{d}t, \quad \alpha > 0.$$

1 Prove that the resolvent satisfies the resolvent equation:

$$U_\alpha - U_\beta = (\beta - \alpha) U_\alpha U_\beta.$$

Hint: it's easiest to start from the expression $U_\alpha U_\beta f$ and then use the definition of the resolvent as well as some properties of the semi-group to rewrite this expression. It's helpful to do a change of variable and then apply Fubini in order to get the result.

2 We claimed in class today that for any f in the range of U_α, $\alpha > 0$, we have that

$$\lim_{t \searrow 0} P_t f = f,$$

where the convergence is uniform. Please verify this.

3 Consider which properties of the probability semi-group we really used to prove that the resolvent satisfies the resolvent equation. Study the proof we (started) in class today (proving that every Feller process defines a probability semi-group) and clarify why this last consideration is relevant.