QED 2015 - Problem Set 1

1. Let $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ be the Minkowski metric. A boost in y-direction with speed v is then described by letting $\beta = \frac{v}{c}, \gamma = \frac{1}{\sqrt{1+\beta^2}}$ and considering

$$t' = \gamma t - \beta\gamma y \tag{1}$$
$$x' = x \tag{2}$$
$$y' = -\beta\gamma t + \gamma y \tag{3}$$
$$z' = z. \tag{4}$$

Let us compute the norm of a four vector x^μ both in the boosted and the unboosted frame:

$$\begin{aligned} x'^\mu x'_\mu &= t'^2 - (x'^2 + y'^2 + z'^2) & (5)\\ &= (\gamma t - \beta\gamma y)^2 - \left(x^2 + (-\beta\gamma t + \gamma y)^2 + z^2\right) & (6)\\ &= (\gamma^2 + \beta^2\gamma^2)t^2 + (\beta^2\gamma^2 - \gamma^2)y^2 + (-2\gamma^2\beta t y + 2\gamma^2\beta)ty - (x^2 + z^2) & (7)\\ &= t^2 - (x^2 + y^2 + z^2) = x^\mu x_\mu, & (8) \end{aligned}$$

where we use $\gamma^2(1+\beta^2) = \frac{1}{1+\beta^2}(1+\beta^2) = 1$ to simplify (7). This shows that the boost doesn't change the four-length of a vector. By means of the identity

$$x^\mu y_\mu = \frac{1}{2}\Big((x+y)^\mu (x+y)_\mu - x^\mu x_\mu - y^\mu y_\mu\Big) \tag{9}$$

we see that the inner product is preserved if and only if the length-square is preserved. So, th inner product in the boosted and unboosted frames are the same: $g_{\mu\nu} = g'_{\mu\nu}$.

2. We start with the mass of the Higgs $m_H \approx 125\,\text{GeV}$ in natural units. Since $c = \hbar = 1$ in natural units ($[\hbar]$ = energy x time, $[c]$ = length x time^{-1}) multiplication with $\hbar^{-1}$ converts an energy into an inverse time and multiplication with c converts a time into a length. So, $125\,\text{GeV}$ is also the corresponding time and length of the Higgs in natural units. To get the results in SI units we have to restore the correct units by multiplication with the right powers of $\hbar$ and c, respectively. We get the following numerical

$$\begin{aligned} m_H &= 125\,\text{GeV}/\text{c}^2 \approx 125 \cdot 10^9 \cdot 1,6 \cdot 10^{-19} \cdot \tfrac{1}{3^2} \cdot 10^{-2 \cdot 8}\text{kg}\\ &\approx 2,2 \cdot 10^{-26}\,\text{kg}\\ t_H &= \tfrac{\hbar}{E_\hbar} = \tfrac{\hbar}{m_H c^2} = \tfrac{1}{125}\,\text{GeV}\,\hbar \approx \frac{10^9}{125} \cdot 1,6 \cdot 10^{-19} \cdot 10^{-34}\text{s}\\ &\approx 1,3 \cdot 10^{-27}\,\text{s}\\ l_H &= t_H c = \tfrac{\hbar}{m_H c} = \tfrac{1}{125}\,\text{GeV}\,\text{c}\hbar \approx \frac{10^9}{125} \cdot 1,6 \cdot 10^{-19} \cdot 10^{-34} \cdot 3 \cdot 10^8\text{m}\\ &\approx 3,8 \cdot 10^{-19}\,\text{m} \end{aligned}$$

3. Let $\psi(x^\mu)$ satisfy the Klein-Gordon equation, i.e.

$$(\partial^\mu \partial_\mu + m^2)\psi = 0. \tag{10}$$

Since $\partial^\mu\partial_\mu$ is a manifestly real differential operator, the complex conjugate ψ^* satisfies the same Klein-Gordon equation. Multiplying equation (10) with ψ^* and ψ respectively, and then subtracting the resulting equations gives

$$\psi^*(\partial^\mu\partial_\mu + m^2)\psi - \psi(\partial^\mu\partial_\mu + m^2)\psi^* = 0. \tag{11}$$

Since $\psi^*\psi = \psi\psi^*$ the mass terms cancel. Now write time and spatial components separately

$$(\psi^*\partial_t^2\psi - \psi\partial_t^2\psi^*) + (\psi^*\nabla^2\psi - \psi\nabla^2\psi^*) = 0. \tag{12}$$

Next we use the product rule $\partial_t(\psi^*\psi_t) = \psi^*\partial_t^2\psi + \partial_t\psi^*\partial_t\psi$ and $\nabla\cdot(\psi^*\nabla\psi) = \psi^*\nabla^2\psi + \nabla\cdot\psi^*\nabla\psi$, respectively the complex conjugated versions thereof, to rewrite each of the four terms. Of the resulting terms, the ones symmetric in ψ and ψ^* cancel, which leaves us with the expression

$$\partial_t(\psi^*\psi_t - \psi\psi_t^*) + \nabla\cdot(\psi^*\nabla\psi - \psi\nabla\psi^*) = 0. \tag{13}$$

Thus defining $\rho := (\psi^*\psi_t - \psi\psi_t^*)$ and $\vec{j} := \psi^*\nabla\psi - \psi\nabla\psi^*$ gives us the desired continuity equation

$$\rho_t + \nabla\cdot\vec{j} = 0 \tag{14}$$

or, in covariant notation, $\partial_\mu j^\mu = 0$ for the four-vector $j^\mu = (c\rho, \vec{j})$.

QED 2015 - Problem Set 2

1. We know

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij}\mathbb{1} \tag{1}$$
$$\{\alpha_i, \beta\} = 0 \tag{2}$$
$$\beta^2 = 1\mathbb{1} \tag{3}$$
$$\gamma^\mu = (\beta, \beta\alpha_i) \tag{4}$$

Using (2) to swap α_i with β on the cost of a sign and using (3) to simplify, we get

$$\begin{aligned}\{\gamma^i, \gamma^j\} &= \{\beta\alpha_i, \beta\alpha_j\} = \beta(\alpha_i\beta)\alpha_j + (\beta\alpha_j)\beta\alpha_i \\ &= -\beta^2\alpha_i\alpha_j - \alpha_j\beta^2\alpha_i = -\{\alpha_i, \alpha_j\} = -2\delta_{i,j}\mathbb{1} \overset{!}{=} 2g^{ij}\mathbb{1}\end{aligned} \tag{5}$$

$$\{\gamma^0, \gamma^j\} = \{\beta, \beta\alpha_j\} = \beta\beta\alpha_j + (\beta\alpha_j)\beta = \beta^2\alpha_j - \alpha_j\beta^2 = 0 \overset{!}{=} 2g^{0j}\mathbb{1} \tag{6}$$

$$\{\gamma^0, \gamma^0\} = \beta^2 + \beta^2 = 2 \cdot \mathbb{1} \overset{!}{=} 2g^{00}\mathbb{1}. \tag{7}$$

This proves $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}$, where $g^{\mu\nu}$ is the (symmetric!) metric tensor of the Minkowski metric with signature $(+---)$.

2. Let $\mu \neq \nu$. Then by the previous exercise, we have $\{\gamma^\mu, \gamma^\nu\} = 0$ and $(\gamma^\nu)^2 = g^{\nu\nu}\mathbb{1}$. Multiplying with γ^ν from the left gives:

$$\gamma^\nu\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\nu\gamma^\mu = 0 \tag{8}$$

Taking traces on both sides and using cyclicity of the trace yields:

$$0 = \mathrm{tr}(\gamma^\nu\gamma^\mu\gamma^\nu) + \mathrm{tr}\big((\gamma^\nu)^2\gamma^\mu\big) \tag{9}$$
$$= 2\mathrm{tr}\big((\gamma^\nu)^2\gamma^\mu\big) \tag{10}$$
$$= 2g^{\nu\nu}\mathrm{tr}(\gamma^\mu). \tag{11}$$

As $g^{\nu\nu} \neq 0$ this implies $\mathrm{tr}\gamma^\mu = 0$.

3. To obtain the "personal representation", observe that $m = 5\,(\mathrm{mod}\,3) = 2$ and $n = 9\,(\mathrm{mod}\,3) = 3$. Thus $U = \mathrm{diag}(\sigma^2, \sigma^3) = U^\dagger$.

$$U\gamma^0U^\dagger = \left(\begin{array}{c|c} \sigma^2 & 0 \\ \hline 0 & \sigma^3 \end{array}\right) \cdot \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -1 \end{array}\right) \cdot \left(\begin{array}{c|c} \sigma^2 & 0 \\ \hline 0 & \sigma^3 \end{array}\right) = \left(\begin{array}{c|c} (\sigma^2)^2 & 0 \\ \hline 0 & -(\sigma^3)^2 \end{array}\right) \tag{12}$$

$$U\gamma^iU^\dagger = \left(\begin{array}{c|c} \sigma^2 & 0 \\ \hline 0 & \sigma^3 \end{array}\right) \cdot \left(\begin{array}{c|c} 0 & \sigma^i \\ \hline -\sigma^i & 0 \end{array}\right) \cdot \left(\begin{array}{c|c} \sigma^2 & 0 \\ \hline 0 & \sigma^3 \end{array}\right) = \left(\begin{array}{c|c} 0 & \sigma^2\sigma^i\sigma^3 \\ \hline -\sigma^3\sigma^i\sigma^2 & 0 \end{array}\right) \tag{13}$$

Using the well-known relation for the Pauli matrices

$$\sigma^i\sigma^j = \delta_{ij}\sigma^0 + i\epsilon_{ijk}\sigma^k \tag{14}$$

we deduce $\sigma^3(\sigma^1\sigma^2) = i\sigma^3\sigma^3 = i\mathbb{1}$ and $\sigma^2\sigma^1\sigma^3 = -i\mathbb{1}$, respectively. Thus

$$U\gamma^0U^\dagger = \gamma^0, \quad U\gamma^1U^\dagger = -i\left(\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right), \quad U\gamma^2U^\dagger = \left(\begin{array}{c|c} 0 & \sigma^3 \\ \hline -\sigma^3 & 0 \end{array}\right), \quad U\gamma^3U^\dagger = \left(\begin{array}{c|c} 0 & \sigma^2 \\ \hline -\sigma^2 & 0 \end{array}\right).$$

So we transformed our vector of matrices γ^μ to the new matrices $\tilde{\gamma}^\mu = (\gamma^0, -i\gamma^5, \gamma^3, \gamma^2)$. Where we have used the notation $\gamma^5 = \left(\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right)$. So, the $\tilde{\gamma}^\mu$'s are in fact just the matrices γ^0 and $-i\gamma^5, \gamma^3, \gamma^2$. The γ^5 matrix is obviously traceless, hermitian and squares to $\mathbb{1}$. So all of the $\tilde{\gamma}^\mu$'s are traceless, $\tilde{\gamma}^0$ being hermitian with eigenvalues ± 1 and the $\tilde{\gamma}^i$'s being anti-hermitian with eigenvalues $\pm i$.

QED 2015 - Problem Set 3

1. Consider the transformation

$$\gamma^\mu \to \tilde{\gamma}^\mu = S(L)^{-1}\gamma^\rho (L^{-1})^\mu{}_\rho S(L), \tag{1}$$

for a Lorentz transformation L. We have to keep in mind that $S(L)$ acts only on Dirac indices (which are suppressed in (1)). For convenience let $\Lambda := L^{-1}$. Then

$$\begin{aligned}
\{\tilde{\gamma}^\mu, \tilde{\gamma}^\nu\} &= \left\{S(L)^{-1}\gamma^\rho \Lambda^\mu{}_\rho S(L), S(L)^{-1}\gamma^\sigma \Lambda^\nu{}_\sigma S(L)\right\} \\
&= S(L)^{-1}\left\{\gamma^\rho \Lambda^\mu{}_\rho, \gamma^\sigma \Lambda^\nu{}_\sigma\right\} S(L) \\
&= S(L)^{-1}\left\{\gamma^\rho, \gamma^\sigma\right\} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma S(L) \\
&= S(L)^{-1}\, 2\cdot g^{\rho\sigma}\, \mathbb{1}\, \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma S(L) \\
&= 2S(L)^{-1}\mathbb{1}S(L)\, \cdot\, \Lambda^\mu{}_\rho g^{\rho\sigma}\Lambda^\nu{}_\sigma \\
&= 2\cdot \mathbb{1} g^{\mu\nu},
\end{aligned} \tag{2}$$

where for the second term in the next to last line, we have used the very definition of a Lorentz transformation (and the fact that the inverse of L is again a Lorentz transformation).

2. Let us calculate the double commutators (we can assume $\rho \neq \sigma$, because otherwise $[\gamma^\rho, \gamma^\sigma] = 0$)

$$[\gamma^\mu, \omega_{\rho\sigma}[\gamma^\rho, \gamma^\sigma]] = \omega_{\rho\sigma}\left(\gamma^\mu\gamma^\rho\gamma^\sigma - \gamma^\rho\gamma^\sigma\gamma^\mu - \gamma^\mu\gamma^\sigma\gamma^\rho + \gamma^\sigma\gamma^\rho\gamma^\mu\right) \tag{3}$$

$$= 2\omega_{\rho\sigma}\left(\gamma^\mu\gamma^\rho\gamma^\sigma - \gamma^\rho\gamma^\sigma\gamma^\mu\right) \tag{4}$$

$$= 2\omega_{\rho\sigma}\left(-\gamma^\rho\gamma^\mu\gamma^\sigma + 2g^{\mu\rho}\gamma^\sigma + \gamma^\rho\gamma^\mu\gamma^\sigma - 2\gamma^\rho g^{\sigma\mu}\right) \tag{5}$$

$$= 4\omega_{\rho\sigma}(\gamma^\sigma g^{\mu\rho} - \gamma^\rho g^{\mu\sigma}) \tag{6}$$

Thus (3) considerably simplifies:

$$[\gamma^\mu, \omega_{\rho\sigma}[\gamma^\rho, \gamma^\sigma]] = 4(\omega^\mu{}_\sigma\gamma^\sigma - \omega_\rho{}^\mu\gamma^\rho). \tag{7}$$

Finally, if $\omega^\mu{}_\nu$ is antisymmetric and we let $S^{\mu\nu} := \frac{i}{4}[\gamma^\mu, \gamma^\nu]$, we get

$$\begin{aligned}
\big[\gamma^\mu, \underbrace{-\tfrac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}}_{=:T}\big] &= -\frac{i}{2}\cdot\frac{i}{4}\cdot 4\cdot 2\cdot \omega^\mu{}_\sigma\gamma^\sigma \\
&= \omega^\mu{}_\nu\gamma^\nu.
\end{aligned} \tag{8}$$

3. Note that the Dirac and the Weyl (chiral) representation only differ in the definition of the timelike component of the gamma-matrix vector:

$$\gamma^0_{\text{Dirac}} = \left(\begin{array}{c|c} \mathbb{1} & 0 \\ \hline 0 & -\mathbb{1} \end{array}\right), \qquad \gamma^0_{\text{Weyl}} = \left(\begin{array}{c|c} 0 & \mathbb{1} \\ \hline \mathbb{1} & 0 \end{array}\right). \tag{9}$$

In the lecture notes we considered a rotation along the $\mathbf{e}_3$-axis by an angle $-\phi$, resulting in the antisymmetric infinitesimal transformation $\omega^j{}_k = -\phi\varepsilon_{3jk}$. Analogously a rotation along the $\mathbf{e}_i$-axis results in the infinitesimal transformation $\omega^j{}_k = -\phi\varepsilon_{ijk}$. To determine the transformation S in spinor space, we calculate

$$\begin{aligned}
\tfrac{4}{i}\cdot S^{12} &= [\gamma^1, \gamma^2] = \gamma^1\gamma^2 - \gamma^1\gamma^2 \\
&= \left(\begin{array}{c|c} 0 & \sigma^1 \\ \hline -\sigma^1 & 0 \end{array}\right)\cdot\left(\begin{array}{c|c} 0 & \sigma^2 \\ \hline -\sigma^2 & 0 \end{array}\right) - \left(\begin{array}{c|c} 0 & \sigma^2 \\ \hline -\sigma^2 & 0 \end{array}\right)\cdot\left(\begin{array}{c|c} 0 & \sigma^1 \\ \hline -\sigma^1 & 0 \end{array}\right) \\
&= \left(\begin{array}{c|c} -[\sigma^1, \sigma^2] & 0 \\ \hline 0 & -[\sigma^1, \sigma^2] \end{array}\right) = -2i\varepsilon_{i12}\left(\begin{array}{c|c} \sigma^i & 0 \\ \hline 0 & \sigma^i \end{array}\right) = -2i(\sigma^3 \oplus \sigma^3),
\end{aligned} \tag{10}$$

where we used the commutation relations for the Pauli matrices $[\sigma^i, \sigma^j] = 2i\,\varepsilon_{ijk}\sigma^k$. Again, this easily generalizes to $\frac{4}{i} \cdot S^{jk} = -2i\varepsilon_{ijk}(\sigma^i \oplus \sigma^i)$. The infinitesimal transformation in spinor space is then given by

$$\begin{aligned} S(R_3) &= \mathbb{1} - (-\phi) \cdot \tfrac{i}{2} \cdot (S^{12} - S^{21}) \qquad (11)\\ &= \mathbb{1} + \phi \cdot \tfrac{i}{2} \cdot \tfrac{i}{4} \cdot (-i)\, 4\, (\sigma^3 \oplus \sigma^3) \\ &= \mathbb{1} + \phi \cdot \tfrac{i}{2}\, (\sigma^3 \oplus \sigma^3), \\ \text{resp. } S(R_i) &= \mathbb{1} + \phi \cdot \tfrac{i}{2}\, (\sigma^i \oplus \sigma^i) \qquad (12) \end{aligned}$$

and the according finite rotation by

$$S(R_i^\phi) = \exp\left(i\frac{\phi}{2}\Sigma_i\right), \qquad (13)$$

where $\Sigma_i = (\sigma^i \oplus \sigma^i)$. A general rotation around the unit vector $\vec{n}$ can be composed by consecutive rotations along x, y and z-axis, which gives rise to the infinitesimal rotation

$$S(R_{\vec{n}}) = \mathbb{1} + i\tfrac{\phi}{2}\, \vec{n} \cdot \vec{\Sigma} \qquad (14)$$

and to the finite rotation in spinor space

$$S(R_{\vec{n}}^\phi) = \exp\left(i\tfrac{\phi}{2}\, \vec{n} \cdot \vec{\Sigma}\right). \qquad (15)$$

QED 2015 - Problem Set 4

1. Consider $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$.

   - Hermitian conjugation results in

$$\begin{aligned}\gamma^{5\dagger} &= -i\gamma^{3\dagger}\gamma^{2\dagger}\gamma^{1\dagger}\gamma^{0\dagger} \\ &= -i(-\gamma^3)(-\gamma^2)(-\gamma^1)\gamma^0 \\ &= i\gamma^3\gamma^2\gamma^1\gamma^0 \\ &= \underbrace{\epsilon_{3210}}_{=1} \cdot i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5. \end{aligned} \tag{1}$$

   - Similarly, squaring gives

$$\begin{aligned}(\gamma^5)^2 &= -(\gamma^0\gamma^1\gamma^2\gamma^3)^2 \\ &= -(\gamma^0\gamma^1\gamma^2\gamma^3)\underbrace{\epsilon_{3210}}_{=1} \cdot (\gamma^3\gamma^2\gamma^1\gamma^0) \\ &= -\mathbb{1}(-\mathbb{1})(-\mathbb{1})(-\mathbb{1}) = \mathbb{1}. \end{aligned} \tag{2}$$

   - For the anticommutator we get

$$\begin{aligned}\{\gamma^5, \gamma^\mu\} &= i\gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3 + i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu \\ &= \gamma^\mu\gamma^5\left(1 + (-1)^3\right) = 0, \end{aligned} \tag{3}$$

     because, when moving γ^μ in the second term from the right to the left, we pick up a minus sign for every transposition with a different γ^ν, which happens exactly three times.

   - Finally, for the vector representation we get

$$[\gamma^5, S^{\mu\nu}] = \frac{i}{4}\left(\gamma^5[\gamma^\mu, \gamma^\nu] - [\gamma^\mu, \gamma^\nu]\gamma^5\right) \tag{4}$$

$$= \frac{i}{4}\left(\gamma^5\gamma^\mu\gamma^\nu - \gamma^5\gamma^\nu\gamma^\mu - \gamma^\mu\gamma^\nu\gamma^5 + \gamma^\nu\gamma^\mu\gamma^5\right) \tag{5}$$

$$= \frac{i}{4}\left(\gamma^5\gamma^\mu\gamma^\nu - \gamma^\mu\gamma^\nu\gamma^5\right) - \frac{i}{4}\left(\gamma^5\gamma^\nu\gamma^\mu - \gamma^\nu\gamma^\mu\gamma^5\right) \tag{6}$$

$$= 0 \tag{7}$$

     because due to the anticommutator relation, by moving γ^5 left or right through two (different) γ^μs we pick up a factor $(-1)^2$. The case $\mu = \nu$ is trivial, because then $S^{\mu\nu} = 0$.

2. Let the matrix $\gamma^5\gamma^0$ act on the equation for a massless spinor $\not{p}\psi = 0$. We rearrange this equation using $p_0 = i\partial_t$ and $p_i = -i\partial_i$

$$i\gamma^5\gamma^0\gamma^\mu\partial_\mu\psi = 0 \tag{8}$$

$$i\gamma^5\gamma^0\gamma^0\partial_0\psi = -i\gamma^5\gamma^0\gamma^i\partial_i\psi \tag{9}$$

$$i\gamma^5\partial_0\psi = -(\gamma^5\gamma^0\gamma^i)i\partial_i\psi \tag{10}$$

$$\gamma^5 p_0\psi = (\gamma^5\gamma^0\gamma^i)p_i\psi. \tag{11}$$

   Next, we inspect the matrices $\gamma^5\gamma^0\gamma^i$. Since

$$\gamma^5\gamma^0 = \left(\begin{array}{c|c} 0 & \mathbb{1} \\ \hline \mathbb{1} & 0 \end{array}\right) \cdot \left(\begin{array}{c|c} \mathbb{1} & 0 \\ \hline 0 & -\mathbb{1} \end{array}\right) = \left(\begin{array}{c|c} 0 & -\mathbb{1} \\ \hline \mathbb{1} & 0 \end{array}\right) \tag{12}$$

   we have

$$\gamma^5\gamma^0\gamma^i = \left(\begin{array}{c|c} 0 & -\mathbb{1} \\ \hline \mathbb{1} & 0 \end{array}\right) \cdot \left(\begin{array}{c|c} 0 & \sigma^i \\ \hline -\sigma^i & 0 \end{array}\right) = \left(\begin{array}{c|c} \sigma^i & 0 \\ \hline 0 & \sigma^i \end{array}\right) = \Sigma^i. \tag{13}$$

   Putting everything together, we get the desired relation between chirality (left hand side) and helicity (right hand side) in the massless case.

$$\gamma^5 p_0\psi = \Sigma^i p_i\psi. \tag{14}$$

3. Let $L \mapsto S(L)$ be a finite dimensional representation of the Lorentz group. To show $\gamma^0 S^\dagger \gamma^0 = S^{-1}$, we first consider infinitesimal transformations $L = \delta^\mu{}_\nu + \varepsilon\omega^\mu{}_\nu$, where the corresponding spinor transformation is known:

$$S = S(L) = \exp\left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right). \tag{15}$$

We thus have

$$\begin{aligned}
\gamma^0 \exp\left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)^\dagger \gamma^0 &= \gamma^0 \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)^{n\dagger}\gamma^0 &(16)\\
&= \gamma^0 \sum_{n=0}^{\infty} \frac{1}{n!}\frac{i}{2}(\omega^*_{\mu\nu})^n (S^{\mu\nu})^{n\dagger}\gamma^0 &(17)
\end{aligned}$$

We have

$$\begin{aligned}
(S^{\mu\nu})^\dagger &= \left(\frac{i}{4}[\gamma^\mu, \gamma^\nu]\right)^\dagger &(18)\\
&= -\frac{i}{4}[\gamma^{\nu\dagger}, \gamma^{\mu\dagger}] = -\frac{i}{4}\gamma^0[\gamma^\nu, \gamma^\mu]\gamma^0 &(19)\\
&= \frac{i}{4}\gamma^0[\gamma^\mu, \gamma^\nu]\gamma^0 = \gamma^0 S^{\mu\nu}\gamma^0, &(20)
\end{aligned}$$

so plugging this in, we get

$$\begin{aligned}
\gamma^0 \exp\left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)^\dagger \gamma^0 &= \gamma^0 \sum_{n=0}^{\infty} \frac{1}{n!}\frac{i}{2}(\omega^*_{\mu\nu})^n (\gamma^0 S^{\mu\nu}\gamma^0)^n\gamma^0 &(21)\\
&= \gamma^0 \sum_{n=0}^{\infty} \frac{1}{n!}\frac{i}{2}(\omega^*_{\mu\nu})^n \gamma^0 (S^{\mu\nu})^n\gamma^0\gamma^0 &(22)\\
&= \sum_{n=0}^{\infty} \frac{1}{n!}\frac{i}{2}\left(\omega^*_{\mu\nu}S^{\mu\nu}\right)^n &(23)\\
&= \exp\underbrace{\left(\frac{i}{2}\omega^*_{\mu\nu}S^{\mu\nu}\right)}_{=A}. &(24)
\end{aligned}$$

On the other hand, for all matrix exponentials $\exp(A)^{-1} = \exp(-A)$, so for infinitesimal transformation with real coefficients, we have shown that $\gamma^0 S^\dagger \gamma^0 = S^{-1}$. Since we consider the Lorentz group as a real Lie group, this is certainly fulfilled.

Finally, we extend our result from infinitesimal transformation to the whole Lorentz group: For proper orthochronous transformations $L \in \mathcal{L}_+^\uparrow$ we can simply iterate the argument for the infinitesimal transformations. To cover the other components of $\mathcal{L}$ we have to show $\gamma^0 \mathrm{P}\gamma^0 = \mathrm{P}^{-1}$ and $\gamma^0\mathrm{T}\gamma^0 = \mathrm{T}^{-1}$, where P stands for parity and T for time reversal.

Since parity corresponds to γ^0, this amounts to showing $\gamma^0\gamma^0\gamma^0 = \gamma^0$ which is trivial. The combination PT corresponds to $S = \gamma^5$. We see that

$$\gamma^0\mathrm{PT}\gamma^0\mathrm{PT} = \gamma^0\gamma^5\gamma^0\gamma^5 = -(\gamma^0)^2\,(\gamma^5)^2 = -\mathbb{1}. \tag{25}$$

So for non-orthochronous transformations $\gamma^0 S^\dagger \gamma^0 = S^{-1}$ is <u>not</u> correct. Instead we pick up an additional sign: $\gamma^0 S^\dagger\gamma^0 = -S^{-1}$. So in general

$$\gamma^0 S(L)^\dagger \gamma^0 = (\mathrm{sgn}\ L^0{}_0)\cdot S(L)^{-1}. \tag{26}$$

QED 2015 - Problem Set 5

1. Let $u(m,0)$ be a spinor such that $(i\partial\!\!\!/ - m)e^{-ikx}u(m,0) = 0$, with $k^\mu = (m,0)$. We already know that $u(m,0) = \binom{\phi}{0}$, where ϕ is an arbitrary Pauli spinor. We see that

$$(k\!\!\!/ - m)\underbrace{\left(\frac{1}{N}(k\!\!\!/ + m)u(m,0)\right)}_{=u(k)} = \frac{1}{N}(k\!\!\!/ - m)(k\!\!\!/ + m)\begin{pmatrix}\phi\\0\end{pmatrix} \tag{1}$$

$$= \frac{1}{N}(k^2 - m^2)\begin{pmatrix}\phi\\0\end{pmatrix} \tag{2}$$

vanishes on-shell for every N and ϕ. First we simplify

$$u(k) = \frac{1}{N}(k\!\!\!/ + m)u(m,0) = \frac{1}{N}(\gamma^\mu k_\mu + m)\begin{pmatrix}\phi\\0\end{pmatrix} \tag{3}$$

$$= \frac{1}{N}\left(\begin{array}{c:c}(E+m)\mathbb{1} & k_i\sigma^i\\ \hdashline -k_i\sigma^i & (-E+m)\mathbb{1}\end{array}\right)\begin{pmatrix}\phi\\0\end{pmatrix} \tag{4}$$

$$= \frac{1}{N}\begin{pmatrix}(E+m)\phi\\ \vec{k}\vec{\sigma}\phi\end{pmatrix}, \tag{5}$$

where we used $k^0 = E$ and $k_i = -k^i$. For $u(k)$ to be a spinor, we must choose N accordingly. Recall the well known identity for the Pauli matrices $(\vec{k}\sigma)^2 = (\vec{k}\cdot\vec{k})\,\mathbb{1}_2$ and the relativistic dispersion relation $m^2 = k^2 = E^2 - (\vec{k}\cdot\vec{k})$.

$$\overline{u}(k)u(k) = \frac{1}{|N|^2}\begin{pmatrix}(E+m)\phi^\dagger & \phi^\dagger(\vec{k}\vec{\sigma})^\dagger\end{pmatrix}\gamma^0\begin{pmatrix}(E+m)\phi\\ \vec{k}\vec{\sigma}\phi\end{pmatrix} \tag{6}$$

$$= \frac{1}{|N|^2}\begin{pmatrix}(E+m)\phi^\dagger & \phi^\dagger\vec{k}\vec{\sigma}\end{pmatrix}\begin{pmatrix}(E+m)\phi\\ -\vec{k}\vec{\sigma}\phi\end{pmatrix} \tag{7}$$

$$= \frac{1}{|N|^2}\left((E+m)^2\phi^\dagger\phi - (\vec{k}\vec{\sigma})^2\phi^\dagger\phi\right) \tag{8}$$

$$= \frac{(E+m)^2 + (m^2 - E^2)}{|N|^2}\phi^\dagger\phi \tag{9}$$

$$= \frac{2mE + 2m^2}{|N|^2}\phi^\dagger\phi \overset{!}{=} \phi^\dagger\phi. \tag{10}$$

Thus choosing $N := \sqrt{2m(m+E)}$ is the right normalization in order to transfer orthogonality properties of the Pauli spinors ϕ to the Dirac spinors $u(k)$. To get the corresponding results for the negative energy spinors $v(k)$, we simply observe that multiplication with γ^5 turns a u-spinor into a v-spinor and vice versa. Since

$$\overline{\gamma^5 u(k)}\gamma^5 u(k) = u(k)^\dagger\gamma^{5\,\dagger}\gamma^0\gamma^5 u(k) = -u(k)^\dagger\gamma^0 u(k) = -\overline{u}(k)u(k), \tag{11}$$

this implies that the same orthonormality relations also hold for v-spinors, with the only exception, that these negative-energy spinors have a *negative* norm-square.

2. Let Λ be a boost in negative x^1-direction with rapidity $\xi > 0$ (so that a particle with vanishing momentum in the original frame has positive momentum in x_1- direction in the transformed frame). We know that the corresponding spinor transformation is given by

$$S(\Lambda) = \begin{pmatrix} \cosh\frac{\xi}{2} & 0 & 0 & \sinh\frac{\xi}{2} \\ 0 & \cosh\frac{\xi}{2} & \sinh\frac{\xi}{2} & 0 \\ 0 & \sinh\frac{\xi}{2} & \cosh\frac{\xi}{2} & 0 \\ \sinh\frac{\xi}{2} & 0 & 0 & \cosh\frac{\xi}{2} \end{pmatrix}. \tag{12}$$

Thus

$$S(\Lambda)u(m,0) = \begin{pmatrix} \cosh\frac{\xi}{2} & 0 & 0 & \sinh\frac{\xi}{2} \\ 0 & \cosh\frac{\xi}{2} & \sinh\frac{\xi}{2} & 0 \\ 0 & \sinh\frac{\xi}{2} & \cosh\frac{\xi}{2} & 0 \\ \sinh\frac{\xi}{2} & 0 & 0 & \cosh\frac{\xi}{2} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \phi_1 \cosh\frac{\xi}{2} \\ \phi_2 \cosh\frac{\xi}{2} \\ \phi_2 \sinh\frac{\xi}{2} \\ \phi_1 \sinh\frac{\xi}{2} \end{pmatrix}. \tag{13}$$

We expect this spinor to coincide with $u(k)$ for $k^\mu = (E; k^1, 0, 0)$, where $k^1 = \pm\sqrt{E^2 - m^2}$. We have to chose the $+$ sign to be in the case of momentum in positive x^1 direction. We can simply check if

$$\frac{1}{N}\begin{pmatrix} (E+m)\phi \\ E\sigma^i\phi \end{pmatrix} = \frac{1}{\sqrt{2m(m+E)}} \begin{pmatrix} \phi_1(E+m) \\ \phi_2(E+m) \\ \phi_2\sqrt{E^2-m^2} \\ \phi_1\sqrt{E^2-m^2}. \end{pmatrix} \tag{14}$$

This is indeed the case once we identify $\cosh\frac{\xi}{2} = \frac{E+m}{\sqrt{2m(m+E)}} > 0$ and $\sinh\frac{\xi}{2} = \frac{\sqrt{E^2-m^2}}{\sqrt{2m(m+E)}} > 0$. This is possible since

$$\cosh^2\tfrac{\xi}{2} - \sinh^2\tfrac{\xi}{2} = \frac{(E+m)^2}{2m(m+E)} - \frac{E^2-m^2}{2m(m+E)} = \frac{2mE+2m^2}{2m(m+E)} = 1. \tag{15}$$

As a sanity check, we can rewrite the energy in terms of the rapidity using

$$\cosh^2\tfrac{\xi}{2} + \sinh^2\tfrac{\xi}{2} = \frac{(E+m)^2}{2m(m+E)} + \frac{E^2-m^2}{2m(m+E)} = \frac{2mE+2E^2}{2m(m+E)} = \frac{E}{m} \tag{16}$$

and the clever hyperbolic identity $\cosh^2\frac{\xi}{2} + \sinh^2\frac{\xi}{2} = \cosh\xi$:

$$E = m\cosh\xi = m + m\frac{v^2}{2} + O(v^2) = E_{\text{rest}} + E_{\text{kin}} + \text{higher order}. \tag{17}$$

3. We consider again the spinors $u(k)$ and $v(k)$ from problem 1. Instead of explicitly prescribing the Pauli spinor ϕ, we choose a basis ϕ_1, ϕ_2 and use superscript indices a, b to denote the Dirac spinors built from e.g. the spin basis $\phi_1 = \binom{1}{0}, \phi_2 = \binom{0}{1}$. The following calculation is similar to the one performed already in problem 1:

$$\begin{aligned} u^{\dagger\,(a)}(k)u^{(b)}(k) &= \frac{1}{|N|^2}\begin{pmatrix} (E+m)\phi_a^\dagger & \phi_a^\dagger(\vec{k}\vec{\sigma})^\dagger \end{pmatrix}\begin{pmatrix} (E+m)\phi_b \\ \vec{k}\vec{\sigma}\phi_b \end{pmatrix} \\ &= \frac{1}{|N|^2}\begin{pmatrix} (E+m)\phi_a^\dagger & \phi_a^\dagger\vec{k}\vec{\sigma} \end{pmatrix}\begin{pmatrix} (E+m)\phi_b \\ \vec{k}\vec{\sigma}\phi_b \end{pmatrix} \\ &= \frac{1}{|N|^2}\left((E+m)^2 + (\vec{k}\vec{\sigma})^2\right)\phi_a^\dagger\phi_b \\ &= \frac{(E+m)^2 + (E^2-m^2)}{2m(E+m)}\phi_b^\dagger\phi_a = \frac{E}{m}\delta^{ab}. \end{aligned} \tag{18}$$

Using again that multiplication with γ^5 turns a u-spinor into a v-spinor, (18) is also true for v-spinors:

$$v^{\dagger\,(a)}(k)v^{(b)}(k) = \left(\gamma^5 u^a(k)\right)^\dagger\left(\gamma^5 u^b(k)\right) = u^{\dagger\,(a)}(k)(\gamma^5)^2 u^{(b)}(k) = u^{\dagger\,(a)}(k)u^{(b)}(k). \tag{19}$$

For the mixed products, the following holds:

$$v^{\dagger\,(a)}(\tilde{k})u^{(b)}(k) = \frac{1}{|N|^2}\begin{pmatrix}\phi_a^\dagger(-\vec{k}\vec{\sigma})^\dagger & (E+m)\phi_a^\dagger\end{pmatrix}\begin{pmatrix}(E+m)\phi_b \\ \vec{k}\vec{\sigma}\phi_b\end{pmatrix} \tag{20}$$

$$= \frac{1}{|N|^2}\left((-\vec{k}\vec{\sigma})(E+m) + (E+m)\,\vec{k}\vec{\sigma}\right)\phi_a^\dagger\phi_b$$

$$= 0, \tag{21}$$

where $\tilde{k}^\mu = (k^0; -\vec{k})$. Furthermore, using the γ^5 trick again, we get

$$u^{\dagger\,(a)}(\tilde{k})v^{(b)}(k) = u^{\dagger\,(a)}(\tilde{k})\gamma^5 u^{(b)}(k) = v^{\dagger\,(a)}(\tilde{k})u^{(b)}(k) = 0. \tag{22}$$

It is now easy to match the coefficients $b(\vec{p}, a)$ and $d^*(\vec{p}, a)$ of a general wave expansion (where $k = (E; \vec{p})$)

$$\psi(t, \vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{m}{E}\sum_a \left\{b(\vec{p}, a)u^a(p)e^{-ik\cdot x} + d^*(\vec{p}, a)v^a(p)e^{ik\cdot x}\right\} \tag{23}$$

to match the initial condition

$$\psi(0, \vec{x}) = \exp\left(-\frac{1}{2}\vec{x}^2 D^{-2}\right) w. \tag{24}$$

Simply let $t = 0$ and multiply (23) from the left with $u^{\dagger(a)}(\vec{p})$ and go to momentum space, to obtain

$$\int d^3x e^{-i\vec{p}\cdot\vec{x}} u^{\dagger(a)}(\vec{p})\psi(0, \vec{x}) = \frac{m}{E}\frac{E}{m}b(\vec{p}, a). \tag{25}$$

where we used the orthogonality relations for u and v. Performing the integral over the Gaussian on the left hand side gives

$$(2\pi D^2)^{\frac{3}{2}} \exp\left(-\frac{1}{2}\vec{p}^2 D^{-2}\right) u^{\dagger(a)}(\vec{p})w = b(\vec{p}, a). \tag{26}$$

In exactly the same way multiplying (23) from the left with $v^{\dagger(a)}(\vec{p})$ gives

$$(2\pi D^2)^{\frac{3}{2}} \exp\left(-\frac{1}{2}\vec{p}^2 D^{-2}\right) v^{\dagger(a)}(\vec{p})w = d^*(\vec{p}, a). \tag{27}$$

QED 2015 - Problem Set 6

1. We start with the kinetic form of the Hamiltonian

$$H_{\text{Pauli}} = \frac{1}{2m}(\vec{\sigma}\cdot\vec{\pi})^2 + eA^0, \tag{1}$$

where $\vec{\pi} = (\vec{p} - e\vec{A})$ is the vector of kinetic momenta. Using the commutation relations and antisymmetry of the ε tensor we get

$$\begin{aligned}
(\vec{\sigma}\cdot\vec{\pi})^2 &= \sigma_i\sigma_j\pi^i\pi^j = (\delta_{ij} + i\varepsilon_{ijk}\sigma_k)\,\pi^i\pi^j && (2)\\
&= \vec{\pi}^2 + i\varepsilon_{ijk}\sigma_k\tfrac{1}{2}[\pi^i,\pi^j] && (3)\\
&= \vec{\pi}^2 + i\varepsilon_{ijk}\sigma_k\tfrac{1}{2}[p^i - eA^i, p^j - eA^j] && (4)\\
&= \vec{\pi}^2 - i\varepsilon_{ijk}\sigma_k e\tfrac{1}{2}\left([p^i, A^j] + [A^i, p^j]\right) && (5)\\
&= \vec{\pi}^2 - i\varepsilon_{ijk}\sigma_k e\tfrac{1}{2}\left(\left\{\tfrac{1}{i}\partial_i A^j + A^j p^i - A^j p^i\right\} + \quad i\leftrightarrow j\quad\right) && (6)\\
&= \vec{\pi}^2 - \varepsilon_{ijk}\sigma_k e\tfrac{1}{2}\left(\partial_i A^j - \partial_j A^i\right) && (7)\\
&= \vec{\pi}^2 - \varepsilon_{ijk}\sigma_k e\partial_i A^j = \vec{\pi}^2 - e\,\mathrm{rot}\vec{A}\cdot\vec{\sigma} && (8)\\
&= \vec{\pi}^2 - e\,\vec{B}\cdot\vec{\sigma}. && (9)
\end{aligned}$$

Thus

$$H_{\text{Pauli}} = \frac{1}{2m}\left(\vec{\pi}^2 - e\,\vec{B}\cdot\vec{\sigma}\right) + eA^0. \tag{10}$$

2. The Dirac equation in the presence of a gauge field is $(i\not{D} - m)\psi = 0$. Acting on this with $(i\not{D} + m)$ gives the equation $(\not{D}^2 - m^2)\psi = 0$, or, written out

$$\begin{aligned}
0 &= \left((i\not{\partial} - e\not{A})^2 - m^2\right)\psi = \left(-\gamma^\mu(\partial_\mu - eA_\mu)\gamma^\nu(\partial_\nu - eA_\nu) - m^2\right)\psi && (11)\\
&= \left(\gamma^\mu\gamma^\nu(\partial_\mu - eA_\mu)(\partial_\nu - eA_\nu) - m^2\right)\psi && (12)\\
&= \left(\gamma^\mu\gamma^\nu\Big\{\underbrace{-\partial_\mu\partial_\nu + e^2A_\mu A_\nu}_{\text{symmetric}} - ie\Big(ip_\mu A_\nu + A_\mu\partial_\nu\Big)\Big\} - m^2\right)\psi && (13)\\
&= \left(\gamma^\mu\gamma^\nu\Big\{\underbrace{-\partial_\mu\partial_\nu + e^2A_\mu A_\nu + ie(A_\nu\partial_\mu + A_\mu\partial_\nu)}_{\text{symmetric}} - ie(\partial_\mu A_\nu)\Big\} - m^2\right)\psi && (14)
\end{aligned}$$

Using that $\gamma^\mu\gamma^\nu = \frac{1}{2}\left(\{\gamma^\mu,\gamma^\nu\} + [\gamma^\mu,\gamma^\nu]\right) = g^{\mu\nu}\mathbb{1} - i\sigma^{\mu\nu}$ and that

$$\sigma^{\mu\nu}\partial_\mu A_\nu = \tfrac{1}{2}\sigma^{\mu\nu}(\partial_\mu A_\nu - \partial_\mu A_\nu) = \tfrac{1}{2}\sigma^{\mu\nu}F_{\mu\nu} \tag{15}$$

we can make use of symmetry/antisymmetry and simplify:

$$\begin{aligned}
0 &= \left((i\not{\partial} - e\not{A})^2 - m^2\right)\psi && (16)\\
&= \left(g^{\mu\nu}\Big\{\dots\Big\} - e\tfrac{i}{2}(2g^{\mu\nu} - 2i\sigma^{\mu\nu})(\partial_\mu A_\nu) - m^2\right)\psi && (17)\\
&= \left((i\partial_\mu - eA_\mu)(i\partial^\nu - eA^\nu) + ieg^{\mu\nu}\partial_\mu A_\nu - ie\left(\partial_\mu A^\mu - i\sigma^{\mu\nu}\partial_\mu A_\nu\right) - m^2\right)\psi && (18)\\
&= \left((i\partial - eA)^2 - \tfrac{e}{2}\sigma^{\mu\nu}F_{\mu\nu} - m^2\right)\psi. && (19)
\end{aligned}$$

3. We separate time- and space-like parts:

$$\tfrac{1}{2}\sigma^{\mu\nu}F_{\mu\nu} = \sigma^{0i}F_{0i} + \tfrac{1}{2}\sigma^{ij}F_{ij} \tag{20}$$

The time-like part can be easily rewritten:

$$\sigma^{0i}F_{0i} = -\sigma^{0i}E_i = \sigma^{0i}E^i = \tfrac{i}{2}[\gamma^0, \gamma^i]E^i \tag{21}$$

$$= i\gamma^0\gamma^i E^i = i\alpha^i E^i = i\vec{\alpha}\cdot\vec{E}, \tag{22}$$

whereas for the space-like part

$$\tfrac{1}{2}\sigma^{ii}F_{ij} = \tfrac{i}{4}[\gamma^i, \gamma^j]\varepsilon_{ijk}B_k = \tfrac{i}{2}\gamma^i\gamma^j\varepsilon_{ijk}B_k \tag{23}$$

$$= \tfrac{i}{2}\varepsilon_{ijk}B_k \left(\begin{array}{c:c} 0 & \sigma^i \\ \hdashline -\sigma^i & 0 \end{array}\right) \cdot \left(\begin{array}{c:c} 0 & \sigma^j \\ \hdashline -\sigma^j & 0 \end{array}\right) \tag{24}$$

$$= -\tfrac{i}{2}\varepsilon_{ijk}B_k \left(\begin{array}{c:c} \sigma^i\sigma^i & 0 \\ \hdashline 0 & \sigma^i\sigma^i \end{array}\right) \tag{25}$$

$$= \tfrac{1}{2} \underbrace{\varepsilon_{ijk}\varepsilon_{ijm}}_{=\delta_{jj}\delta_{km}-\delta_{jm}\delta_{kj}} B_k \left(\begin{array}{c:c} \sigma^m & 0 \\ \hdashline 0 & \sigma^m \end{array}\right) \tag{26}$$

$$= \tfrac{1}{2}\delta_{km}B_k \left(\begin{array}{c:c} \sigma^m & 0 \\ \hdashline 0 & \sigma^m \end{array}\right) \tag{27}$$

$$= B_k\Sigma^k = -B^k\Sigma^k = -\vec{B}\cdot\vec{\Sigma}. \tag{28}$$

Finally we have

$$\tfrac{1}{2}\sigma^{\mu\nu}F_{\mu\nu} = i\vec{\alpha}\cdot\vec{E} - \vec{B}\cdot\vec{\Sigma}. \tag{29}$$

While the physical interpretation for the second term is fairly obvious (the magnetic field couples to the spin resulting in the potential energy of a magnetic moment), the meaning of the first term is not so clear (at least to me - the obvious guess: an electric dipole contribution cannot be true, since e^- has no measurable dipole moment).

QED 2015 - Problem Set 7

1. In this problem we calculate the commutator of H_{Dirac} with angular momentum resp. spin

   (a)

$$\begin{aligned}
[H_D, L_i] &= [\vec{\alpha}\cdot\vec{p}, L_i] = [\vec{\alpha}\cdot\vec{p}, (\vec{x}\times\vec{p})_i] && (1)\\
&= \varepsilon_{ijk}[\vec{\alpha}\cdot\vec{p}, x_j p_k] && (2)\\
&= \vec{\alpha}\cdot\varepsilon_{ijk}\left([\vec{p}, x_j]p_k + x_j[\vec{p}, p_k]\right) && (3)\\
&= \varepsilon_{ijk}\,\underbrace{\vec{\alpha}\cdot[\vec{p}, x_j]}_{=-i\alpha^j}\,p_k && (4)\\
&= -i(\vec{\alpha}\times\vec{p})_i && (5)
\end{aligned}$$

   where we have used the canonical commutation relation.

   (b)

$$\begin{aligned}
[H_D, S_i] &= \tfrac{1}{2}[\vec{\alpha}\cdot\vec{p} + \beta m, \Sigma_i] = \tfrac{1}{2}[\vec{\alpha}\cdot\vec{p} + \beta m, \gamma^5\alpha^i] && (6)\\
&= \tfrac{1}{2}[\gamma^0\gamma^j p_j + m\gamma^0, \gamma^5\gamma^0\gamma^i] && (7)\\
&= \tfrac{1}{2}p_j\left(\gamma^0\gamma^j\gamma^5\gamma^0\gamma^i - \gamma^5\gamma^0\gamma^i\gamma^0\gamma^j + m(\gamma^0\gamma^5\gamma^0\gamma^i - \gamma^5\gamma^0\gamma^i\gamma^0)\right) && (8)\\
&= \tfrac{1}{2}p_j\gamma^5\left(-\gamma^j\gamma^i + \gamma^i\gamma^j + m(-\gamma^i + \gamma^i)\right) && (9)\\
&= \tfrac{1}{2}p_j\gamma^5[\gamma^i, \gamma^j] && (10)
\end{aligned}$$

   where we have used $\vec{\Sigma} = \vec{\alpha}\gamma^5$ and the anticommutation relations for the gamma matrices. A quick computation shows

$$[\gamma^i, \gamma^j] = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} = -\begin{pmatrix} [\sigma^i, \sigma^j] & 0 \\ 0 & [\sigma^i, \sigma^j] \end{pmatrix}.$$

   Since γ^5 turns this block-diagonal matrix into the corresponding off-diagonal matrix, we finally have (using the well-known commutation relation for the Pauli-matrices)

$$\begin{aligned}
[H_D, S_i] &= -2i\tfrac{1}{2}p_j\varepsilon_{ijk}\begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix} = -i\varepsilon_{ijk}p_j\alpha^k && (11)\\
&= -i(\vec{p}\times\vec{\alpha})_i. && (12)
\end{aligned}$$

2. First we compute

   (a)

$$\begin{aligned}
[\vec{\sigma}\cdot\vec{x}, J_i] &= [\vec{\sigma}\cdot\vec{x}, L_i + \tfrac{1}{2}\Sigma_i] && (13)\\
&= \vec{\sigma}\cdot[\vec{x}, L_i] + \tfrac{1}{2}[\vec{\sigma}, \Sigma_i]\cdot\vec{x} && (14)\\
&= \vec{\sigma}\cdot[\vec{x}, (x\times p)_i] + \tfrac{1}{2}[\vec{\sigma}, \sigma_i]\cdot\vec{x} && (15)\\
&= \sigma^j\varepsilon_{ikm}\underbrace{[x_j, x_k p_m]}_{=i\delta_{jm}x_k} + i\varepsilon_{ris}x_r\sigma^s && (16)\\
&= i\sigma^j\varepsilon_{ikj}x_k + i\varepsilon_{ris}x_r\sigma^s, && (17)
\end{aligned}$$

   where we used the fact, that we are dealing with Pauli spinors and $\vec{\Sigma} = \vec{\sigma}\oplus\vec{\sigma}$. Upon relabeling the dummy index r by k and the dummy index s by j we get

$$[\vec{\sigma}\cdot\vec{x}, J_i] = i\sigma^j x_k(\varepsilon_{ikj} + \varepsilon_{kij}) = 0. \qquad (18)$$

This relation is not changed if we multiply $\vec{x}$ by any operator in position space of the form $f(r)$ since then:

$$[\vec{\sigma}\cdot\vec{x}f(r), J_i] = [\vec{\sigma}\cdot\vec{x}, J_i]f(r) + (\vec{\sigma}\cdot\vec{x})\underbrace{[f(r), J_i]}_{=0} = 0. \tag{19}$$

(b) To show

$$0 = \{\vec{\sigma}\cdot\vec{x}, \vec{\sigma}\cdot\vec{L}+1\} = \{\vec{\sigma}\cdot\vec{x}, \vec{\sigma}\cdot\vec{L}\} + 2(\vec{\sigma}\cdot\vec{x}), \tag{20}$$

we compute

$$\begin{aligned}
\{\vec{\sigma}\cdot\vec{x}, \vec{\sigma}\cdot\vec{L}\} &= \{\sigma^i x_i, \sigma^j L_j\} && (21)\\
&= \sigma^i x_i \sigma^j L_j + \sigma^j L_j \sigma^i x_i && (22)\\
&= \left(\sigma^i\sigma^j x_i L_j + \sigma^j\sigma^i x_i L_j\right) + \left(\sigma^j\sigma^i L_j x_i - \sigma^j\sigma^i x_i L_j\right) && (23)\\
&= \{\sigma^i, \sigma^j\} x_i L_j + \sigma^j\sigma^i [L_j, x_i] && (24)\\
&= \tfrac{1}{2}\left(\{\sigma^i, \sigma^j\}\{x_i, L_j\} + [\sigma^j, \sigma^i][L_j, x_i]\right) && (25)\\
&= \tfrac{1}{2}\left(2\delta^{ij}\{x_i, L_j\} + (2i\varepsilon_{jin}\sigma^n)(i\varepsilon_{ijk}x_k)\right) && (26)\\
&= \delta^{ij}\{x_i, L_j\} - \varepsilon_{ijn}\sigma^n\varepsilon_{ijk}x_k) && (27)\\
&= \delta^{ij}\{x_i, L_j\} - 2\delta_{kn}\sigma^n x_k && (28)\\
&= -2\vec{\sigma}\vec{x}, && (29)
\end{aligned}$$

because

$$\delta^{ij}\{x_i, L_j\} = \vec{x}\cdot\vec{L} + \vec{L}\cdot\vec{x} = \varepsilon_{ijk}(x_i x_j p_k + x_j p_k x_i) = 0 \tag{30}$$

due to the antisymmetricity of the epsilon tensor and the canonical commutation relations. Again, this relation is not changed if we multiply $\vec{x}$ by any operator in position space of the form $f(r)$:

$$\{\vec{\sigma}\cdot\vec{x}f(r), \vec{\sigma}\cdot\vec{L}+1\} = \{\vec{\sigma}\cdot\vec{x}, \vec{\sigma}\cdot\vec{L}+1\}f(r) - (\vec{\sigma}\cdot\vec{x})[f(r), \vec{\sigma}\cdot\vec{L}+1], \tag{31}$$

using the rule $\{A, BC\} = \{A, B\}C - B[A, C]$. Clearly

$$[f(r), \vec{\sigma}\cdot\vec{L}+1] = [f(r), \vec{\sigma}\cdot\vec{L}] = \vec{\sigma}\cdot[f(r), \vec{L}] = 0. \tag{32}$$

(c) Since position operator and Pauli-matrices act in orthogonal subspaces we immediately get

$$(\vec{\sigma}\cdot\hat{\vec{x}})^2 = \sigma^i\hat{\vec{x}}_i\sigma^j\hat{\vec{x}}_j = \sigma^i\sigma^j\hat{\vec{x}}_i\hat{\vec{x}}_j = \tfrac{1}{2}\{\sigma^i, \sigma^j\}\hat{\vec{x}}_i\hat{\vec{x}}_j = \delta^{ij}\hat{\vec{x}}_i\hat{\vec{x}}_j = \mathbb{1}_2\otimes\mathbb{1}, \tag{33}$$

i.e. $\vec{\sigma}\cdot\hat{\vec{x}}$ is an involution. As for any involution all eigenvalues have to square to one, i.e. $+1$ or -1.

Let ϕ^+_{jm}, ϕ^-_{jm} be the spinor harmonics with $\vec{J}^2$ eigenvalue $j(j+1)$, J_z eigenvalue m and positive resp. negative eigenvalue of K. Due to the block-structure of all involved operators we deal only with the Pauli-spinors ϕ^+_{jm}, ϕ^-_{jm} (whereas we really should use the Dirac spinors $\phi^+_{jm}\oplus\phi^-_{jm}$ resp. $\phi^-_{jm}\oplus\phi^+_{jm}$.) We show

(A) $(\vec{\sigma}\cdot\hat{\vec{x}})\phi^\pm_{jm}$ has again $\vec{J}^2$ eigenvalue $j(j+1)$ and J_z eigenvalue m,

(B) $(\vec{\sigma}\cdot\hat{\vec{x}})\phi^\pm_{jm}$ has K eigenvalue $-k$,

(C) $(\vec{\sigma}\cdot\hat{\vec{x}})\phi^\pm_{jm}$ is again a spinor harmonic.

ad A:

$$
\begin{aligned}
\vec{J}^2(\vec{\sigma}\cdot\hat{\vec{x}})\phi^{\pm}_{jm} &= \sum_i J_i^2(\vec{\sigma}\cdot\hat{\vec{x}})\phi^{\pm}_{jm} && (34)\\
&= \sum_i J_i\Big((\vec{\sigma}\cdot\hat{\vec{x}})J_i + \overbrace{[J_i,(\vec{\sigma}\cdot\hat{\vec{x}})]}^{=0}\Big) && (35)\\
&= \sum_i (\vec{\sigma}\cdot\hat{\vec{x}})J_i^2\phi^{\pm}_{jm} = j(j+1)\phi^{\pm}_{jm}, && (36)
\end{aligned}
$$

$$
\begin{aligned}
J_z(\vec{\sigma}\cdot\hat{\vec{x}})\phi^{\pm}_{jm} &= (\vec{\sigma}\cdot\hat{\vec{x}})J_z + \overbrace{[J_z,(\vec{\sigma}\cdot\hat{\vec{x}})]}^{=0} = m\phi^{\pm}_{jm} && (37)\\
& && (38)
\end{aligned}
$$

ad B:

$$
\begin{aligned}
K(\vec{\sigma}\cdot\hat{\vec{x}})\phi^{\pm}_{jm} &= (\vec{\sigma}\cdot\vec{L}+1)(\vec{\sigma}\cdot\hat{\vec{x}})\phi^{\pm}_{jm} && (39)\\
&= \tfrac{1}{2}\Big([\vec{\sigma}\cdot\vec{L}+1,\vec{\sigma}\cdot\hat{\vec{x}}] + \underbrace{\{\vec{\sigma}\cdot\vec{L}+1,\vec{\sigma}\cdot\hat{\vec{x}}\}}_{=0}\Big)\phi^{\pm}_{jm} && (40)\\
&= \tfrac{1}{2}K(\vec{\sigma}\cdot\hat{\vec{x}})\phi^{\pm}_{jm} - \tfrac{1}{2}(\vec{\sigma}\cdot\hat{\vec{x}})K\phi^{\pm}_{jm} && (41)\\
&= \tfrac{1}{2}K(\vec{\sigma}\cdot\hat{\vec{x}})\phi^{\pm}_{jm} - \tfrac{1}{2}k(\vec{\sigma}\cdot\hat{\vec{x}})\phi^{\pm}_{jm}, && (42)
\end{aligned}
$$

implying $K(\vec{\sigma}\cdot\hat{\vec{x}})\phi^{\pm}_{jm} = -k(\vec{\sigma}\cdot\hat{\vec{x}})\phi^{\pm}_{jm}$.

ad C: From classical QM we know that $|jm\rangle$ constitute a set of non degenerate states. When going to two-component spinors the possible degeneracy of such states becomes 2. The quantum number k lifts this degeneracy, such that each state $|jmk\rangle$ is unique up to a phase. Thus we already know from (c) that

$$
(\vec{\sigma}\cdot\hat{\vec{x}})\phi^{\pm}_{jm} = \eta\phi^{\mp}_{jm} \tag{43}
$$

with $\eta=-1$ or $\eta=1$. To determine the correct η we have to inspect only one, preferably simple, spinor harmonic. Let's take an s-state with spin up, i.e. $j=m=\frac{1}{2}$ and $k>0$:

$$
\begin{aligned}
(\vec{\sigma}\cdot\hat{\vec{x}})\phi^{+}_{1/2,1/2} &= (\vec{\sigma}\cdot\hat{\vec{x}})\frac{1}{\sqrt{2\cdot 0+1}}\begin{pmatrix}\sqrt{0+1/2+1/2}\;Y_{0,0}\\ \sqrt{0-1/2+1/2}\;Y_{0,1}\end{pmatrix} = \frac{1}{2\sqrt{\pi}}(\vec{\sigma}\cdot\hat{\vec{x}})\begin{pmatrix}1\\0\end{pmatrix} && (44)\\
&= \frac{1}{2\sqrt{\pi}|x|}\begin{pmatrix}x_3\\ x_1+ix_2\end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix}Y_{10}\\ -Y_{11}\end{pmatrix}, && (45)
\end{aligned}
$$

which coincides with the spinor $\phi^{-}_{1/2,1/2}$ coming from the p-state. Thus $\eta=1$ is indeed the correct sign.

QED 2015 - Problem Set 8

1. The spin operator is defined as $\frac{1}{2}\Sigma_i = \frac{i}{8}\varepsilon_{ijk}[\gamma^j, \gamma^k]$. We compute the sum of squares $(\vec{\Sigma})^2$:

$$\begin{aligned}
\Sigma_0^2 + \Sigma_2^2 + \Sigma_3^2 &= (\tfrac{i}{4})^2 \sum_i \varepsilon_{ijk}\varepsilon_{inm}[\gamma^j, \gamma^k][\gamma^n, \gamma^m] && (1)\\
&= -\tfrac{1}{16}(\delta_{jn}\delta_{km} - \delta_{jm}\delta_{kn})[\gamma^j, \gamma^k][\gamma^n, \gamma^m] && (2)\\
&= -\tfrac{1}{16}\sum_{j,k}\Big([\gamma^j, \gamma^k][\gamma^j, \gamma^k] - [\gamma^j, \gamma^k][\gamma^k, \gamma^j]\Big) && (3)\\
&= -\tfrac{1}{8}\sum_{j,k}[\gamma^j, \gamma^k]^2 && (4)\\
&= -\tfrac{1}{8}\sum_{j\neq k}(\gamma^j\gamma^k - \gamma^k\gamma^j)^2 && (5)\\
&= -\tfrac{1}{8}\sum_{j\neq k}(2\gamma^j\gamma^k)^2 && (6)\\
&= -\tfrac{1}{2}\sum_{j\neq k}\gamma^j\gamma^k\gamma^j\gamma^k && (7)\\
&= \tfrac{1}{2}\sum_{j\neq k}\gamma^j\gamma^k\gamma^k\gamma^j && (8)\\
&= \tfrac{1}{2}\sum_{j\neq k}(-\mathbb{1})(-\mathbb{1}) && (9)\\
&= \tfrac{1}{2}(3\cdot 3 - 3)\mathbb{1} = 3\cdot\mathbb{1}, && (10)
\end{aligned}$$

where along with the usual identities for the gamma matrices, we have used the contracted epsilon identity in line (2).

2. Plugging into the radial equations

$$\left(E - m + \frac{Ze^2}{4\pi r}\right) g(r) + f'(r) + \tfrac{k}{r} f(r) = 0, \qquad (11)$$

$$\left(E + m + \frac{Ze^2}{4\pi r}\right) f(r) - g'(r) + \tfrac{k}{r} g(r) = 0 \qquad (12)$$

the ansatz

$$f(r) = \sqrt{1 - \tfrac{E}{m}}\, e^{-\lambda r}(2\lambda r)^\gamma \sum_{n=0}^{\infty}(a_n - b_n)(2\lambda r)^n, \qquad (13)$$

$$g(r) = \sqrt{1 + \tfrac{E}{m}}\, e^{-\lambda r}(2\lambda r)^\gamma \sum_{n=0}^{\infty}(a_n + b_n)(2\lambda r)^n, \qquad (14)$$

gives the following lowest order terms in $(2\lambda r)^{\gamma-1}$:

$$\frac{Ze^2}{4\pi}\sqrt{1 + \tfrac{E}{m}}2\lambda(a_0 + b_0) + \sqrt{1 - \tfrac{E}{m}}2\lambda\gamma(a_0 - b_0) + k\sqrt{1 - \tfrac{E}{m}}2\lambda(a_0 - b_0) = 0, \qquad (15)$$

$$\frac{Ze^2}{4\pi}\sqrt{1 - \tfrac{E}{m}}2\lambda(a_0 - b_0) - \sqrt{1 + \tfrac{E}{m}}2\lambda\gamma(a_0 + b_0) + k\sqrt{1 + \tfrac{E}{m}}2\lambda(a_0 + b_0) = 0. \qquad (16)$$

Multiplying with $\frac{1}{2}\lambda^{-1}\sqrt{1 \mp \frac{E}{m}}$ thus gives:

$$\frac{Ze^2}{4\pi}\sqrt{1 - \tfrac{E^2}{m^2}}(a_0 + b_0) + \left(1 - \tfrac{E}{m}\right)(k + \gamma)(a_0 - b_0) = 0, \qquad (17)$$

$$\frac{Ze^2}{4\pi}\sqrt{1 - \tfrac{E^2}{m^2}}(a_0 - b_0) + \left(1 + \tfrac{E}{m}\right)(k - \gamma)(a_0 + b_0) = 0. \qquad (18)$$

This is a system of two linear equations in $A := (a_0 + b_0)$ and $B := (a_0 - b_0)$. Wlog. we can assume $a_0 \neq 0$ or $b_0 \neq 0$, because otherwise we could replace in our ansatz γ by $\gamma + 1$ until we reach coefficients a_n, b_n, where at least one them doesn't vanish. Therefore also A and B cannot both vanish, i.e. we are looking for a non trivial solution of our linear equations. This means that the determinant of the coefficient-matrix

$$\begin{pmatrix} \frac{Ze^2}{4\pi}\sqrt{1-\frac{E^2}{m^2}} & \left(1-\frac{E}{m}\right)(k+\gamma) \\ \left(1+\frac{E}{m}\right)(k-\gamma) & \frac{Ze^2}{4\pi}\sqrt{1-\frac{E^2}{m^2}} \end{pmatrix} \tag{19}$$

must vanish! This gives us the nice equation

$$\left(\frac{Ze^2}{4\pi}\right)^2 \left(1-\tfrac{E^2}{m^2}\right) = \left(1-\tfrac{E^2}{m^2}\right)(k^2-\gamma^2) \tag{20}$$

Cancelling and regrouping yields

$$\gamma^2 = k^2 - \left(\frac{Ze^2}{4\pi}\right)^2, \tag{21}$$

or, taking the positive root and using $\alpha = \frac{e^2}{4\pi}$,

$$\gamma = \sqrt{k^2 - Z^2\alpha^2}. \tag{22}$$

QED 2015 - Problem Set 9

1. We look for an unitary transformation C in spinor-space which satisfies

$$\gamma^\mu = -C(\gamma^\mu)^T C^{-1}, \tag{1}$$

where $\mu = 0, 1, \ldots d$ is the spin $\frac{1}{2}$ representation of the Lorentz group $O(d,1)$ in d-dimensional space. Transposing equation (1) gives

$$(\gamma^\mu)^T = -(C^{-1})^T \gamma^\mu C^T, \tag{2}$$

which can be reinserted into (1):

$$\gamma^\mu = C(C^{-1})^T \gamma^\mu C^T C^{-1}, \tag{3}$$

which implies $[C(C^{-1})^T, \gamma^\mu] = 0$ for all $\mu = 0, 1, \ldots d$. Thus we can invoke Schur's lemma, which due to the irreducibility of the spin $\frac{1}{2}$ representation implies that $C(C^{-1})^T$ is a multiple of the identity, i.e.

$$C = \lambda C^T, \tag{4}$$

for some nonzero constant $\lambda \in \mathbb{C}$. Iterating

$$C = \lambda C^T = \lambda(\lambda C^T)^T = \lambda^2 C, \tag{5}$$

restricts λ to ± 1, i.e. C is unitary and either symmetric or antisymmetric. Taking determinants implies $\det C = \lambda^{d+1} \det C$. So for $d+1$ odd, only $\lambda = 1$, i.e. symmetric C is possible.

2. Charge conjugation affects a spinor ψ as follows

$$\psi \to \psi^c = C\overline{\psi}^T = C(\gamma^0)^T \psi^* \tag{6}$$

$$\overline{\psi} \to \overline{\psi}^c = \overline{\psi}^* C^\dagger \gamma^0 \tag{7}$$

Since

$$\begin{aligned}
\gamma^5 &= i\gamma^0\gamma^1\gamma^2\gamma^3 && (8)\\
&= i(-C(\gamma^0)^T C^{-1})(-C(\gamma^1)^T C^{-1})(-C(\gamma^2)^T C^{-1})(-C(\gamma^3)^T C^{-1}) && (9)\\
&= iC(\gamma^0)^T(\gamma^1)^T(\gamma^2)^T(\gamma^3)^T C^{-1} && (10)\\
&= i\varepsilon_{3210} C(\gamma^0\gamma^1\gamma^2\gamma^3)^T C^{-1} && (11)\\
&= C(\gamma^5)^T C^{-1} && (12)
\end{aligned}$$

we have the relation $CP_L^T = P_L C$ for the chiral projection. So we can easily compute the action of charge conjugation on, say, the left handed chiral subspace

$$\begin{aligned}
P_L\psi \to (P_L\psi)^c &= C\overline{(P_L\psi)}^T && (13)\\
&= C\left(\psi^\dagger P_L^\dagger \gamma^0\right)^T && (14)\\
&= C(\gamma^0)^T P_L^T \psi^* && (15)\\
&= -\gamma^0 C P_L^T \psi^* && (16)\\
&= -\gamma^0 P_L C \psi^* && (17)\\
&= -P_R \gamma^0 C \psi^* && (18)\\
&= P_R C(\gamma^0)^T \psi^* && (19)\\
&= P_R(\psi^c), && (20)
\end{aligned}$$

i.e. charge conjugation flips chirality (where we have used $\{\gamma^0, \gamma^5\} = 0$ to get from line (17) to line (18)). To determine the action of charge conjugation on vectors rather than spinors we consider $\Gamma^\mu = \overline{\psi}\gamma^\mu\psi$, which we know transforms as a vector:

$$\begin{aligned}
\Gamma^\mu \to (\Gamma^\mu)^c &= \overline{\psi}^c \gamma^\mu \psi^c && (21)\\
&= (\overline{\psi}^* C^\dagger \gamma^0)\gamma^\mu (C(\gamma^0)^T \psi^*) && (22)\\
&= g^{\mu\mu} \left(\overline{\psi}[C^T (\gamma^0)^T (\gamma^\mu)^T (C^\dagger)^T \gamma^0]\psi\right)^* && (23)\\
&= g^{\mu\mu} \left(\overline{\psi}[\gamma^0 \gamma^\mu \gamma^0]\psi\right)^* && (24)\\
&= \left(\overline{\psi}\gamma^\mu\psi\right)^* = (\Gamma^\mu)^*, && (25)
\end{aligned}$$

which is very confusing since, on the four-potential we want charge conjugation to act as $A^\mu \xrightarrow{C} -A^\mu$.

Let us now turn to parity. For parity we know the action on vectors: $(t, \vec{x}) \to (t, -\vec{x})$. So we can reverse the above reasoning to infer the parity transformation for spinors.

$$\begin{aligned}
\Gamma^\mu \to (\Gamma^\mu)^p &= g^{\mu\mu}\,\Gamma^\mu && (26)\\
&\overset{!}{=} \overline{\psi}^P \gamma^\mu \psi^P && (27)\\
&= \overline{\psi}[P^\dagger \gamma^0 \gamma^\mu P]\psi. && (28)
\end{aligned}$$

So P acts on spinor space subject to the condition to the rule $\gamma^0\gamma^\mu P = g^{\mu\mu} P = P\gamma^\mu\gamma^0$. Thus $P = \gamma^0$ is (up to a phase) the parity transformation on spinor space. From this we infer $PP_R = \gamma_0 P_R = P_L\gamma^0 = P_L P$, similar to the charge conjugation.

The total action on a spinor is

$$\psi(t, \vec{x}) \xrightarrow{P} \psi^P(t, -\vec{x}) \xrightarrow{C} \psi^{CP}(t, -\vec{x}) = C(\gamma^0)^T(\gamma^0)^*\psi^*(t, -\vec{x}) = C\psi^*(t, -\vec{x}) \tag{29}$$

For a left handed spinor we have $(P_L\psi)^c = P_R(\psi^c)$ for charge conjugation and likewise $(P_R\psi)^p = P_L(\psi^p)$ for a right handed spinor under parity transformation. So (29) becomes

$$P_L\psi(t, \vec{x}) \xrightarrow{P} P_R\psi^P(t, -\vec{x}) \xrightarrow{C} P_L\psi^{CP}(t, -\vec{x}) = P_L C\psi^*(t, -\vec{x}) \tag{30}$$

i.e. chirality is preserved under CP.

QED 2015 - Problem Set 10

1. Consider the Hamiltonian for the conjugate classical fields $\phi(x), \pi(x)$:

$$H = \int d^3x \; \pi^*(x)\pi(x) + \nabla\phi^*(x)\nabla\phi(x) + m^2\phi^*(x)\phi(x). \tag{1}$$

Introducing the Fourier-transformed field we get

$$\int d^3x \; \pi^*(x)\pi(x) = \int d^3x \, d^3k \, d^3k' \; e^{i(\vec{k}-\vec{k}')x}\tilde{\pi}^*(\vec{k}')\tilde{\pi}(\vec{k}) \tag{2}$$

$$= \frac{1}{(2\pi)^3}\int d^3k \, d^3k' \; \delta(\vec{k}-\vec{k}')\tilde{\pi}^*(\vec{k}')\tilde{\pi}(\vec{k}) \tag{3}$$

$$= \frac{1}{(2\pi)^3}\int d^3k \; \tilde{\pi}^*(\vec{k})\tilde{\pi}(\vec{k}), \tag{4}$$

$$\int d^3x \; \phi^*(x)\phi(x) = \frac{1}{(2\pi)^3}\int d^3k \; \tilde{\phi}^*(\vec{k})\tilde{\phi}(\vec{k}), \tag{5}$$

$$\int d^3x \; \nabla\phi^*(x)\nabla\phi(x) = \frac{1}{(2\pi)^3}\int d^3k \; i\vec{k}(-i\vec{k})\tilde{\phi}^*(\vec{k})\tilde{\phi}(\vec{k}). \tag{6}$$

Thus

$$H = \frac{1}{(2\pi)^3}\int d^3k \; \tilde{\pi}^*(\vec{k})\tilde{\pi}(\vec{k}) + (m^2+\vec{k}^2)\tilde{\phi}^*(\vec{k})\tilde{\phi}(\vec{k}). \tag{7}$$

Merging terms and using the relativistic dispersion relation $k^2 = k_0^2 - \vec{k}^2 = m^2$ we get

$$H = \frac{1}{(2\pi)^3}\int d^3k \; |\tilde{\pi}(\vec{k})|^2 + k_0^2|\tilde{\phi}(\vec{k})|^2. \tag{8}$$

2. The quantized Hamiltonian for a scalar field is

$$H = \int d^3x \; \left(\pi^\dagger\pi + \nabla\phi^\dagger\nabla + m^2\phi^\dagger\phi\right). \tag{9}$$

The field operators can be written in momentum space as

$$\phi = \frac{1}{(2\pi)^3}\int d^3k \; \frac{1}{2\omega_k}\left(a(k)e^{i\vec{k}\cdot\vec{x}} + b^\dagger(k)e^{-i\vec{k}\cdot\vec{x}}\right), \tag{10}$$

$$\pi = \frac{1}{(2\pi)^3}\int d^3k \; \frac{1}{2i}\left(b(k)e^{i\vec{k}\cdot\vec{x}} - a^\dagger(k)e^{-i\vec{k}\cdot\vec{x}}\right). \tag{11}$$

Thus (using the substitution $k \to -k$ resp. $k' \to -k'$ where convenient)

$$\phi^\dagger\phi = \frac{1}{(2\pi)^6}\int\int \frac{d^3k \; d^3k'}{4\omega_k\omega_{k'}}\left(b(k)e^{i\vec{k}\cdot\vec{x}} + a^\dagger(k)e^{-i\vec{k}\cdot\vec{x}}\right)\left(a(k')e^{i\vec{k}'\cdot\vec{x}} + b^\dagger(k')e^{-i\vec{k}'\cdot\vec{x}}\right), \tag{12}$$

$$= \frac{1}{(2\pi)^6}\int\int \frac{d^3k \; d^3k'}{4\omega_k\omega_{k'}}e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}\left(b(k)b^\dagger(k') + a^\dagger(-k)a(-k')\right) \tag{13}$$

$$+ \frac{1}{(2\pi)^6}\int\int \frac{d^3k \; d^3k'}{4\omega_k\omega_{k'}}e^{i(\vec{k}+\vec{k}')\cdot\vec{x}}\left(b(k)a(k') + a^\dagger(-k)b^\dagger(-k')\right). \tag{14}$$

Due to the canonical commutation relation for ϕ and π the second integral on the right hand-side contains only combinations of a and b that vanish. The first integral on can be further rewritten into

$$\phi^\dagger\phi = \frac{1}{(2\pi)^6}\int\int \frac{d^3k \; d^3k'}{4\omega_k\omega_{k'}}e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}\left(b(k)b^\dagger(k') + a^\dagger(k)a(k')\right). \tag{15}$$

Similarly we get

$$\pi^\dagger\pi = \frac{1}{(2\pi)^6}\int\int \frac{d^3k\ d^3k'}{4} e^{i(\vec{k}-\vec{k}')\cdot\vec{x}} \left(b(k)b^\dagger(k') + a^\dagger(k)a(k')\right). \tag{16}$$

Finally, applying ∇ to the field ϕ brings down the exponential factor, so

$$\nabla\phi^\dagger\nabla\phi = \frac{1}{(2\pi)^6}\int\int \frac{d^3k\ d^3k'}{4\omega_k\omega_{k'}} e^{i(\vec{k}-\vec{k}')\cdot\vec{x}} \left(i\vec{k}b(k)(-i\vec{k}')b^\dagger(k') + (-i\vec{k})a^\dagger(k)i\vec{k}'a(k')\right) \tag{17}$$

$$= \frac{1}{(2\pi)^6}\int\int \frac{d^3k\ d^3k'}{4\omega_k\omega_{k'}} \vec{k}\vec{k}' e^{i(\vec{k}-\vec{k}')\cdot\vec{x}} \left(b(k)b^\dagger(k') + a^\dagger(k)a(k')\right) \tag{18}$$

Integration with respect to x each results in a δ function and so eliminates thus the k'-integration. We then have

$$\int d^3x\ \nabla\phi^\dagger\nabla\phi = \frac{1}{(2\pi)^3}\int \frac{d^3k}{4\omega_k^2}\vec{k}^2 \left(b(k)b^\dagger(k) + a^\dagger(k)a(k)\right), \tag{19}$$

$$\int d^3x\ m^2\phi^\dagger\phi = \frac{m^2}{(2\pi)^3}\int \frac{d^3k}{4\omega_k^2}m^2 \left(b(k)b^\dagger(k) + a^\dagger(k)a(k)\right), \tag{20}$$

$$\int d^3x\ \pi^\dagger\pi = \frac{1}{(2\pi)^3}\int \frac{d^3k}{4} \left(b(k)b^\dagger(k) + a^\dagger(k)a(k)\right). \tag{21}$$

Finally, we can use the commutation relations $[a, a^\dagger] = [b, b^\dagger]$ to write

$$H = \int d^3\tilde{k}\ \left(\frac{\vec{k}^2 + m^2}{2\omega_k} + \frac{\omega_k}{2}\right) \left(b(k)b^\dagger(k) + a^\dagger(k)a(k)\right) \tag{22}$$

$$= \int d^3\tilde{k}\ \omega_k \left(b(k)b^\dagger(k) + a^\dagger(k)a(k)\right) \tag{23}$$

$$= \frac{1}{2}\int d^3\tilde{k}\ \omega_k \left(b(k)^\dagger b(k) + b(k)b^\dagger(k) + \underbrace{[b, b^\dagger] + [a^\dagger, a]}_{=0} + a^\dagger(k)a(k) + a(k)a^\dagger(k)\right). \tag{24}$$

Applying normal ordering, again gives

$$: H : = \int d^3\tilde{k}\ \omega_k \left(N_b + N_a\right). \tag{25}$$

QED 2015 - Problem Set 11

1. The field operators for a scalar field are given by

$$\phi(t', \vec{x}') = \int d\tilde{k}' \left(a(k')e^{-ik'x'} + b^\dagger(k')e^{ik'x'} \right), \tag{1}$$

$$\phi^\dagger i(t, \vec{x}) = \int d\tilde{k} \left(b(k)e^{-ikx} + a^\dagger(k)e^{ikx} \right). \tag{2}$$

Thus, when forming the normal ordered product $:\phi\phi^\dagger:$ only the term containing $aa^\dagger$ is the wrong order. Therefore

$$:\phi(t', \vec{x}')\phi^\dagger(t, \vec{x}): = \phi(t', \vec{x}')\phi^\dagger(t, \vec{x}) + \int\int d\tilde{k}\, d\tilde{k}'[a^\dagger(k), a(k')]e^{i(kx-k'x')} \tag{3}$$

$$= \phi(t', \vec{x}')\phi^\dagger(t, \vec{x}) + \int\int d\tilde{k}\; e^{ik(x-x')} \tag{4}$$

$$= \phi(t', \vec{x}')\phi^\dagger(t, \vec{x}) + \langle 0|\phi(t', \vec{x}')\phi^\dagger(t, \vec{x})|0\rangle, \tag{5}$$

where in the last line we have used the explicit form of the vacuum expectation value. Finally, observe that (3) is also valid if we replace $\phi\phi^\dagger$ with $\mathsf{T}\phi\phi^\dagger$ and that first time-ordering and then normal-ordering is the same as just normal-ordering. So,

$$:\phi(t', \vec{x}')\phi^\dagger(t, \vec{x}): = \mathsf{T}\phi(t', \vec{x}')\phi^\dagger(t, \vec{x}) + \langle 0|\mathsf{T}\phi(t', \vec{x}')\phi^\dagger(t, \vec{x})|0\rangle. \tag{6}$$

2. On a classical level, the Lagrangian $\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$ yields the conjugate momenta $\pi_\psi = i\bar{\psi}\gamma^0$ resp. $\pi_{\bar{\psi}} = 0$. Thus the classical Hamiltonian is given by

$$\mathcal{H} = i\bar{\psi}\gamma^0\dot{\psi} - i\bar{\psi}i\partial\!\!\!/\psi + m\bar{\psi}\psi. \tag{7}$$

Using the expansion for each 4-spinor component

$$\psi_\alpha(x) = \int \tilde{d}k \left(b_a(k)u^a_\alpha(k)e^{-ikx} + d^\dagger_a(k)v^a_\alpha(k)e^{ikx} \right), \tag{8}$$

$$\bar{\psi}_\alpha(x) = \int \tilde{d}k \left(d_a(k)\bar{v}^a_\alpha(k)e^{-ikx} + b^\dagger_a(k)\bar{u}^a_\alpha(k)e^{ikx} \right), \tag{9}$$

where inside the integrals one has to sum over the 2-spinor index $a = 1, 2$, we could evaluate, one after another, all three terms of the Hamiltonian. Instead, we first try to compute the time derivative $\dot{\psi}_\alpha(x)$:

$$i\partial_t\psi_\alpha(x) = \int \tilde{d}k\; \omega_k \left(b_a(k)u^a_\alpha(k)e^{-ikx} - d^\dagger_a(k)v^a_\alpha(k)e^{ikx} \right). \tag{10}$$

The Heisenberg equation of motion implies $i\partial_t\psi_\alpha = [\psi_\alpha, :H:]$, i.e. we have to identify the right hand side of (10) as commutator with the Hamiltonian.

In analogy to the case of scalar field, we try a Hamiltonian of the form

$$:H: = \int\int \tilde{d}k\; \omega_k \sum_{a=1,2} \left(N^{(a)}_b + N^{(a)}_d \right). \tag{11}$$

To compute the commutator $[\psi_\alpha, {:}H{:}]$, we therefore have to evaluate terms of the following form:

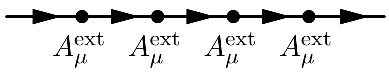
$$\begin{aligned}
[b_{a'}(k'), b_a^\dagger(k) b_a(k)] &= \{b_{a'}(k'), b_a^\dagger(k)\} b_a(k) - b_a^\dagger(k)\{b_{a'}(k'), b_a(k)\} \\
&= (2\pi)^3 \frac{\omega_k}{m} \delta_{aa'} \delta(\vec{k} - \vec{k}') b_a(k), \qquad (12) \\
[b_{a'}(k'), d_a^\dagger(k) d_a(k)] &= \{b_{a'}(k'), d_a^\dagger(k)\} d_a(k) - d_a^\dagger(k)\{b_{a'}(k'), d_a(k)\} \\
&= 0, \qquad (13) \\
[d_{a'}^\dagger(k'), b_a^\dagger(k) b_a(k)] &= \{d_{a'}^\dagger(k'), b_a^\dagger(k)\} b_a(k) - b_a^\dagger(k)\{d_{a'}^\dagger(k'), b_a(k)\} \\
&= 0. \qquad (14) \\
[d_{a'}^\dagger(k'), d_a^\dagger(k) d_a(k)] &= \{d_{a'}^\dagger(k'), d_a^\dagger(k)\} d_a(k) - d_a^\dagger(k)\{d_{a'}^\dagger(k'), d_a(k)\} \\
&= -(2\pi)^3 \frac{\omega_k}{m} \delta_{aa'} \delta(\vec{k} - \vec{k}') d_a(k), \qquad (15)
\end{aligned}$$

where we have made extensive use of the commutator-anticommutator formula $[A, BC] = \{A, B\}C - B\{A, C\}$, as well as of the canonical anticommutation relations. When integrating each of the expressions (12) to (15) all the prefactors nicely cancel with the integration measure $\tilde{dk}$. Since only (12) survives with a plus-sign and (15) with a minus-sign, the commutator $[\psi_\alpha, {:}H{:}]$ indeed coincides with (10). This means that, subject to the canonical anticommutation relations, we have "guessed" the Hamiltonian correctly.

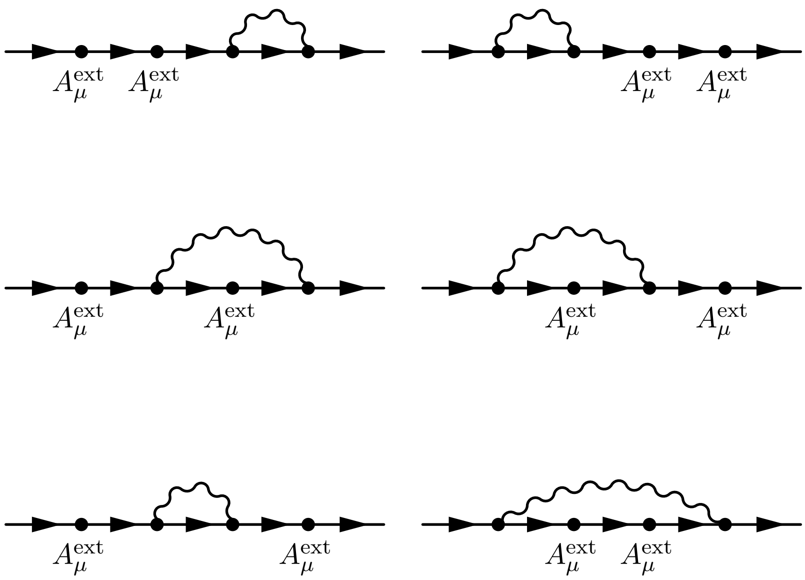
QED 2015 - Problem Set 12

1. Connected Feynman diagrams of order e^4 have an even number of internal vertices, i.e. photons can be paired only with internal vertices. So we can order all diagrams by first by the number of exchanged photons and then by the number of fermionic loops:

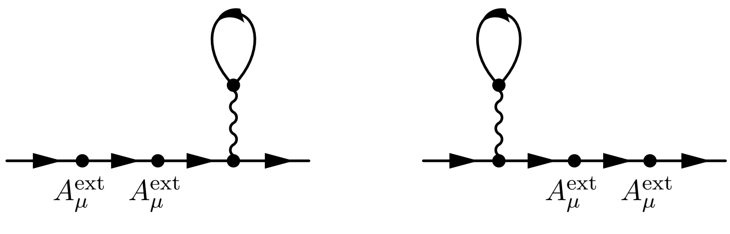
   (a) No photons, four interactions with the external field.



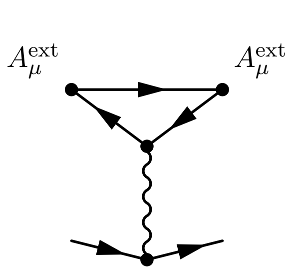
   (b) One photon, two interactions with the external field and no loops.



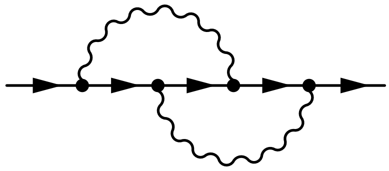
   (c) Same as above, but with one loop correction.



   (d) Same as above, but with one vertex correction.

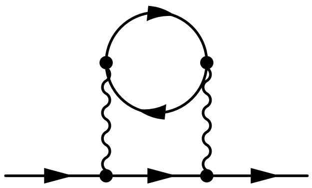


(e) Two photons, no interactions with the external field.

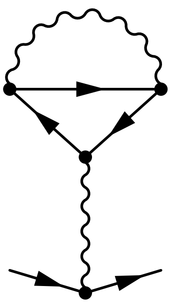


plus the rest of the $\binom{4}{2} = 6$ possible arrangements, cf. (b)

(f) Same as (e), but with one loop.



(g) Same as (1e), but with one vertex correction.



2. Let's have a closer look at the abouve diagram from (1g): We have one photon propagator, two external interactions and one incoming resp. outgoing electron propagator. At each vertex, momentum conservation must hold. The incoming (and outgoing) momenum p is fixed. So are the interactions with the external field. Thus the diagram corresponds to the term

$$iS_F(p)(-ie\gamma^\mu)iS_F(p)\tfrac{1}{i}G_F(p)\int \frac{d^4p'}{(2\pi)^4}(-ie\gamma^\mu)iS_F(p')(-ie\not{A}^{\text{ext}})iS_F(p'')(-ie\not{A}^{\text{ext}})iS_F(p-p'-p''),$$

which carries both a Minkowski index μ and two (suppressed) spinor indices α, β. If the external interaction A^{ext} is not constant in momentum space, one has to additionally perform a k-integration for each occurrence of A^{ext}.

3. The integral in problem 2 contains three photon propagators, each of which is proportional (in absolute value) to p^{-1}. Thus the integral is proportional to $\int dp\, p$ which is quadratically divergent.