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2. Exercise on QFT for many-body systems

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Intro: The Fermi liquid (FL) theory by Lev Landau, postulating the existence of independent quasi-particles with the same charge and spin of the original electrons but with energy $\epsilon_{\mathbf{p}} \neq \frac{p^2}{2m}$, allows to solve the "puzzle" of the metallic physics, that is to understand why the qualitative behavior of several observables in metals resembles so closely that of a non-interacting (!) electron gas. The classical (historical) starting point is the phenomenological consideration that the energy change δE due to adding/removal of quasiparticles ($\delta n_{\mathbf{p},\sigma}$) with momentum $\mathbf{p}$ and spin σ to/from the Fermi sphere is given by the following functional (note that in the truly non-interacting case, this would just be a number) :

$$\delta E[\delta n_{\mathbf{p},\sigma}] = \sum_{\mathbf{p},\sigma} \tilde{\epsilon}_{\mathbf{p}}\, \delta n_{\mathbf{p},\sigma} + \frac{1}{2} \sum_{\mathbf{p},\mathbf{p}';\,\sigma,\sigma'} f_{\sigma,\sigma'}(\mathbf{p},\mathbf{p}')\, \delta n_{\mathbf{p},\sigma}\, \delta n_{\mathbf{p}',\sigma'} + \cdots \tag{1}$$

The coefficients of the first term of the sum ($\tilde{\epsilon}_{\mathbf{p}}$) , which represents the energies for creating an excitation with momentum $\mathbf{p}$ <u>without</u> considering the feedback effect of the other quasiparticles, are usually expanded (in the isotropic case) as $\tilde{\epsilon}_{\mathbf{p}} \sim \tilde{\epsilon}_F + \tilde{v}_F(p - p_F) + \cdots$, whereas $v_F = |\frac{\partial \tilde{\epsilon}_{\mathbf{p}}}{\partial \mathbf{p}}_{|p=p_F}| \simeq \frac{p_F}{m^*}$, being m^* the (enhanced) effective mass[1]. Finally, it is important to note that the quasi-particle distribution function $n_{\mathbf{p}}$ has the same form of as for non-interacting electrons, but in term of the quasiparticle energy $\epsilon_{\mathbf{p}}$.

3. First steps in calculating a Fermi liquid (FL)

1+1+2=3+1* points*

a) Verify that the so-called quasiparticle density of states for $\epsilon = \tilde{\epsilon}_F$ defined as $\tilde{N}(\epsilon) = \frac{1}{L^d}\sum_{\mathbf{p}} \delta(\epsilon - \tilde{\epsilon}_{\mathbf{p}})$ can be easily expressed as $\frac{m^*}{m}\, N(\epsilon_F)$, where $N(\epsilon_F)$ is the DOS of the corresponding non-interacting system.

b) Derive from Eq. 1, the formal expression of the (full) quasi-particle energy, defined as the energy necessary to add an excitation of momentum $\mathbf{p}$ close to the Fermi level, that is $\epsilon_{\mathbf{p}} = \frac{\delta E}{\delta n_{\mathbf{p},\sigma}}$. Which is the physical meaning of the term correcting the value of $\tilde{\epsilon}_{\mathbf{p}}$? Are the values of $\epsilon_{\mathbf{p}}$ depending on temperature or chemical potential? Motivate your answer.

c) Calculate the specific heat at constant volume c_V for the non-interacting Fermi gas in three dimensions:

$$c_V = \left(\frac{\partial E}{\partial T}\right)_V$$

where for this specific case

$$E = E_{\text{kin}} = \left\langle \sum_{\vec{p},\sigma} \epsilon_p c^\dagger_{\vec{p},\sigma} c_{\vec{p},\sigma} \right\rangle \quad \text{with } \epsilon_p = \frac{p^2}{2m}$$

What is the temperature dependence of c_V? How will the final result change for interacting electrons under the assumption that the Fermi liquid theory can be applied?

[1] As it will be discussed in the next Lectures and Exercises, the mass enhancement can be related microscopically to the momentum/frequency derivatives of the self-energy at the Fermi level

4. How to sum over Matsubara frequencies *3+2+2=7 points*

As it was discussed in the Lecture, the fermionic Green function in imaginary time reads

$$G(\mathbf{k},\tau) = -\frac{1}{Z}\mathrm{Tr}\left\{\mathrm{e}^{-\beta\mathcal{H}}\mathrm{T}\left[c_{\mathbf{k}}(\tau)c^{\dagger}_{\mathbf{k}}(0)\right]\right\} \tag{2}$$

being Z the partition function, β the inverse temperature and T is the imaginary-time ordering operator. When transforming this expression to frequency space

$$G(\mathbf{k},\tau) = \frac{1}{\beta}\sum_{n}\mathrm{e}^{-i\omega_n\tau}G(\mathbf{k},i\omega_n), \tag{3}$$

using the cyclic properties of the trace, one can immediately deduce that the sum in Eq. 3 has to be performed only over the discrete so-called fermionic Matsubara frequencies $\omega_n = \frac{\pi}{\beta}(2n+1)$. When performing the explicit evaluation of Feynman diagrams in terms of physical quantities, a typical intermediate step consists exactly of this evaluation of sums over Matsubara frequencies. We will consider here the simplest cases, which represent, however, the basis for performing more complicate calculations occurring in realistic situations.

The particle density $\langle n\rangle$ of an electronic system can be expressed in term of the Green function as follows

$$\langle n\rangle = \frac{1}{L^d}\sum_{\mathbf{k}}\langle c^{\dagger}_{\mathbf{k}}c_{\mathbf{k}}\rangle = \frac{1}{L^d}\sum_{\mathbf{k}}G(\mathbf{k},\tau=0^{-}) \tag{4}$$

$$= \frac{1}{L^d}\sum_{\mathbf{k}}\frac{1}{\beta}\sum_{n}\mathrm{e}^{-i\omega_n 0^{-}}G(\mathbf{k},i\omega_n) \tag{5}$$

a) Perform the Matsubara sum in (5) for the case of non-interacting electrons with energy dispersion $\epsilon_{\mathbf{k}}$, whose Green function is given by $G(\mathbf{k},i\omega_n)=\frac{1}{i\omega_n-\epsilon_{\mathbf{k}}}$ [*Hint: note that $i\omega_n$ are exactly the simple poles of the Fermi distribution function in the complex plane with residue $-\beta^{-1}$. This means that the Matsubara sum can be written as an integral over a contour enclosing all Matsubara frequencies. Note also that the Green function is analytic in the lower/upper complex half-plane. Exploiting these analytic properties of the integrand, it is convenient to further transform this contour into two disconnected contours extending in the whole complex plane.*]

b) Think about a possible numerical implementation of Eq. (5), e.g. suppose one knows the value of $G(\mathbf{k},i\omega_n)$ for a finite set of frequencies (say from $-i\omega_M$ to $i\omega_M$) . What would be wrong with a "straightforward" numerical evaluation of such expression (i.e., just summing up all values available)? Suggest possible tricks to correct the problems encountered and to get reliable numerical results.

c) Often one has to calculate so-called "bubble" diagrams of the form

$$\frac{1}{L^d}\sum_{\mathbf{k}}\frac{1}{\beta}\sum_{n}G(\mathbf{k},i\omega_n)G(\mathbf{k}+\mathbf{q},i\omega_n+i\Omega_m) \tag{6}$$

where Ω_m is an "external" bosonic Matsubara frequency given by $\Omega_m = 2m\pi/\beta$. Using the free particle case again, calculate the expression (6) analytically, and discuss explicitly the results for the two limiting cases (i) $\Omega_m = 0, \mathbf{q}\to 0$ ("static limit"), and (ii) $\mathbf{q}=0, \Omega_m \to 0$ ("dynamic limit"). [*Hint: a convenient way of proceeding is to use a partial fraction decomposition.*]

* Bonus points

Viel Spaß!