Digital Communications 1

Written exam on November 25, 2013

Institute of Telecommunications

Vienna University of Technology

Please note:

- You may use the official lecture notes, a pocket calculator, and a collection of mathematical formulas.
- Personal notes, materials from exercise classes, and pre-calculated problems may not be used.
- Legible writing and a clear layout of your derivations and solutions are absolutely necessary!
- Provide detailed derivations. When using results from the lecture notes, they must be explicitly referenced.

Problem 1 (20 credits)

Consider a passband PAM system with an ML sequence detector. In the equivalent discrete-time baseband domain, the channel is described by the folded spectrum

$$S_h(z) = \frac{3z^2 + 10z + 3}{36z}.$$

Furthermore, the channel adds white noise.

a) Sketch the function $S_h(e^{j\theta})$ for $\theta \in [-\pi, \pi]$.

b) Find the poles and zeros of $S_h(z)$.

c) Find a minimum phase factorization of $S_h(z)$.

d) Calculate the transfer function and impulse response of the equivalent discrete-time system including the noise whitening filter.

e) Consider the transmission of statistically independent symbols $a[k] \in \{1 + 2j, 1 - 2j, -1 + 2j, -1 - 2j\}$ with equal probabilities. Assume that the Viterbi algorithm is to be implemented at the output of the noise whitening filter. Sketch the state transition diagram and one elementary stage of the trellis diagram.

Problem 2 (20 credits)

A random variable $S \in \{s^{(i)}\}_{i=1}^{4}$ with

$$s^{(1)} = -2, \quad s^{(2)} = -1, \quad s^{(3)} = 1, \quad s^{(4)} = 2$$

is distributed according to

$$p_I(1) = \frac{1}{7}, \quad p_I(2) = \frac{2}{7}, \quad p_I(3) = \frac{3}{7}.$$

S is corrupted by additive noise N, yielding the observation

$$Y = S + N\,.$$

Here, N is statistically independent of S and distributed according to the Cauchy distribution:

$$f_N(n) = \frac{1/\pi}{n^2+1}\,.$$

a) Calculate the probabilities $P\{N > n_1\}$ and $P\{N < n_2\}$. *Hint*: $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$.

b) Calculate the likelihood function $f_{Y|I}(y|i)$ and the posterior distribution $p_{I|Y}(i|y)$.

c) Calculate and sketch the MAP decision rule $\hat{\imath}_{\mathrm{MAP}}(y)$.

d) Calculate the conditional error probability of the MAP detector for each $i \in \{1, \ldots, 4\}$.

e) Calculate the unconditional error probability of the MAP detector.

Problem 3 (20 credits)

A transmission system uses the two transmit signals

$$s_1(t) = \sin(\omega_0 t)\ \mathrm{rect}(t - T; T)$$
$$s_2(t) = \sin(\omega_0 t + \theta)\ \mathrm{rect}(t - T; T),$$

with $T = \pi/\omega_0$ and $\omega_0 > 0$.

a) Calculate $\langle s_1, s_2 \rangle$ and sketch it as a function of θ.

b) Calculate $\|s_1\|$, $\|s_2\|$, and $\|s_1 - s_2\|$.

Consider transmission of $s_i(t)$ with $i \in \{1, 2\}$ over an AWGN channel:

$$y(t) = s_i(t) + n(t),$$

where the noise $n(t)$ has a power spectral density of $N_0/2$.

c) Find the decision rule of the ML detector $\hat{\imath}_{\mathrm{ML}}(y)$. Sketch an implementation of the ML detector that uses only one inner product.

d) Using your results from b), calculate the error probability of the ML detector.

Problem 4 (20 credits)

Consider transmission of a symbol $A \in \{1, 2\}$. At the slicer input, the receiver observes a random variable Q that is corrupted by real-valued multiplicative noise Z:

$$Q = A\,Z\,.$$

The noise is Laplace distributed, i.e.,

$$f_Z(z) = e^{-2\,|z-1|}\,.$$

a) For each $a \in \{1, 2\}$, calculate and sketch $f_{Q|A}(q|a)$.

b) Calculate the probabilities $\mathrm{P}\{Q > \eta_1 \,|\, A = 1\}$ and $\mathrm{P}\{Q < \eta_2 \,|\, A = 2\}$ for $\eta_1 > 1$ and $\eta_2 < 2$.

c) Consider ML detection. Calculate the decision rule and sketch the decision regions. Calculate the conditional symbol error probability $\mathrm{P}\{\mathcal{E} \,|\, A = a\}$ for each $a \in \{1, 2\}$.

d) Consider a detector using the following decision rule:

$$\hat{A} = \begin{cases} 1\,, & \big|Q-1\big| < 0.5 \\ 2\,, & \big|Q-2\big| < 0.5 \\ \text{undecided}, & \text{otherwise.} \end{cases}$$

Sketch the decision regions. Calculate the conditional symbol error probabilities $\mathrm{P}\{\hat{A} = 1 \,|\, A = 2\}$ and $\mathrm{P}\{\hat{A} = 2 \,|\, A = 1\}$.