# Modulations- und Detektionsverfahren

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Institute of Communications and Radio-Frequency Engineering

Vienna University of Technology

**Please note:**

- You may use the official lecture notes, a pocket calculator, and a collection of mathematical formulas.
- Personal notes, materials from exercise classes, and pre-calculated problems may not be used.
- Legible writing and a clear layout of your derivations and solutions are absolutely necessary!
- Provide detailed derivations. When using results from the lecture notes, they must be explicitly referenced.

**Problem 1 (20 credits)**

A random variable $S$ from the symbol alphabet $\mathcal{S} \subset \mathbb{C}$ is corrupted by additive noise $N \in \mathbb{C}$ that is statistically independent of $S$. The real part and imaginary part of the noise, $N_\mathrm{R}$ and $N_\mathrm{I}$, are statistically independent and have Laplacian distributions

$$f_{N_\mathrm{R}}(n_\mathrm{R}) = \frac{1}{2}\,e^{-|n_\mathrm{R}|} \qquad f_{N_\mathrm{I}}(n_\mathrm{I}) = \frac{1}{2}\,e^{-|n_\mathrm{I}|}.$$

**a)** What is the ML decision criterion? (*Hint:* Split $S$ and the observed variable $Y$ into their real and imaginary parts: $S = S_\mathrm{R} + jS_\mathrm{I}$, $Y = Y_\mathrm{R} + jY_\mathrm{I}$.)

**b)** Sketch the signal constellation and the ML decision boundaries for the following signal alphabets:

**b1)** $\mathcal{S} = \{1+j, 1-j\}$

**b2)** $\mathcal{S} = \{1+j, -1+j\}$

**b3)** $\mathcal{S} = \{1+j, -1-j\}$

**c)** Now assume $\mathcal{S} = \{1+j, 1-j, -1+j, -1-j\}$. Sketch the signal constellation and the ML decision region for detecting $\hat{S}_\mathrm{ML} = 1+j$. Assuming that $S = 1+j$ was transmitted, calculate the probability of a correct ML detection.

**d)** Using the result from c), calculate the conditional error probabilities $P_\mathrm{ML}\{\mathcal{E}|S\}$ for all $S \in \mathcal{S}$ and the unconditional error probability $P_\mathrm{ML}\{\mathcal{E}\}$.

**Problem 2 (20 credits)**

Binary symbols $a[k] \in \{1, -1\}$ are transmitted by means of passband PAM over a dispersive channel with additive noise. The transmit symbols are equally likely. The receiver consists of a receive filter, a symbol-rate sampler, and a slicer. The equivalent baseband overall pulse is

$$p(t) = A\left(1 - \frac{t^2}{\tau^2}\right) \mathrm{rect}\left(t; \frac{4\tau}{3}\right), \qquad \text{with } \tau = 10\text{ms}, \; A > 0.$$

**a)** Sketch the overall pulse $p(t)$ and determine the maximum bit rate for ISI-free transmission and the corresponding symbol duration $T_S$. This bit rate is to be used in what follows.

**b)** How does the output $q[k]$ of the symbol-rate sampler depend on the transmit symbols $a[k]$?

**c)** At the slicer input, the filtered discrete-time noise is zero-mean, white, and circularly symmetric complex Gaussian with variance $N_0$. The slicer uses the following decision rule:

$$\hat{a}[k] = \begin{cases} 1\,, & \mathrm{Re}\{q[k]\} > 0 \\ -1\,, & \mathrm{Re}\{q[k]\} < 0. \end{cases}$$

What is the symbol error probability of the receiver?

**Problem 3 (20 credits)**

For two complex signals $x(t)$ and $y(t)$, consider the quantities $\|x\|$, $\|y\|$, $\langle x, y\rangle$, and $\|x - y\|$.

**a)** Show how these quantities are changed by the following transformations of *both* signals $x(t)$ and $y(t)$:

**a1)** a time shift by $t_0$;

**a2)** a frequency shift by $\omega_0$;

**a3)** multiplication by a constant factor $\alpha \in \mathbb{C}$.

**b)** Consider the signals

$$x(t) = A\cos(\omega_0 t)\ \mathrm{rect}(t; T), \qquad y(t) = A\cos(\omega_0 t + \theta)\ \mathrm{rect}(t; T),$$

with $T = 2\pi/\omega_0$. (*Note:* It is $\mathrm{rect}(t; T)$, not $\mathrm{rect}(t; T/2)$.)

**b1)** Calculate $\|x\|$, $\|y\|$, $\langle x, y\rangle$, and $\|x - y\|$.

**b2)** Sketch these quantities as a function of $\theta$.

**Problem 4 (20 credits)**

Consider equalization of a channel using a decision feedback equalizer. The equivalent discrete-time baseband pulse is given by

$$p[k] = \delta[k] + \frac{1}{2}\delta[k-1] - \frac{1}{2}\delta[k+1].$$

The transmit symbols are taken from the alphabet $\{2, -2\}$ with equal probabilities. The symbol sequence and the equivalent discrete-time noise $Z[k]$ are uncorrelated and both white. The noise is zero-mean, and the noise variance is $\sigma_Z^2 = 1/4$.

**a)** Assume a general feedforward filter $d[k]$, $k \in [-L, L]$. Calculate the equivalent overall pulse $p^{(d)}[k]$ at the output of the feedforward filter. How long is this pulse?

**b)** Consider now the case $L = 1$. What is the minimum length $K$ of the feedback filter such that as much ISI as possible is canceled (under the assumption that all previous symbol decisions were correct)?

**c)** The coefficients of the feedforward filter are $\mathbf{d}_{\mathrm{MSE}} = (8/689)\cdot(21\ 50\ -41)^T$. Calculate the coefficients of the feedback filter (of length $K$, as calculated in b)) which minimize the MSE at the slicer input.

**d)** Specify the signal at the slicer input and indicate the components which constitute 1) the desired signal, 2) ISI, and 3) additive noise (still assuming that all previous symbol decisions were correct).

**e)** Calculate the mean powers of the three signal components from d).