Digital Communications 1
Written exam on January 26, 2015

Institute of Telecommunications

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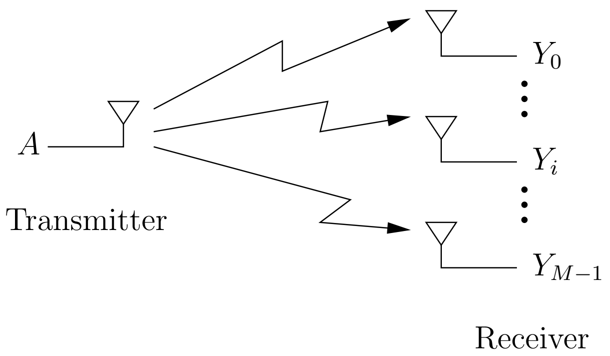
Please note:

- You may use the official lecture notes, a pocket calculator, and a collection of mathematical formulas.
- Personal notes, materials from exercise classes, and pre-calculated problems may not be used.
- Legible writing and a clear layout of your derivations and solutions are absolutely necessary!
- Provide detailed derivations. When using results from the lecture notes, they must be explicitly referenced.

Problem 1 (20 credits)

To mitigate fading effects in a mobile communication system, multiple receive antennas are considered (see figure):



The discrete-time channel associated with the ith receive antenna is given by $Y_i = H_i\, A + N_i$, $i = 0, 1, \ldots, M-1$. Here, $A \in \{0, 1\}$ is the transmit symbol ($P[A = 0] = 1/2$), Y_i is the receive symbol, H_i is a fading coefficient, and N_i is additive noise (all quantities are real-valued). The overall input-output relation can be summarized as

$$\mathbf{Y} = A\mathbf{H} + \mathbf{N}\,,$$

where $\mathbf{Y} = (Y_0 \ldots Y_{M-1})^T$, $\mathbf{H} = (H_0 \ldots H_{M-1})^T$, and $\mathbf{N} = (N_0 \ldots N_{M-1})^T$. Both the channel coefficients and the noise are Gaussian with mean $\mathrm{E}\{\mathbf{H}\} = \mathrm{E}\{\mathbf{N}\} = \mathbf{0}$ and correlation matrices $\mathrm{E}\{\mathbf{H}\,\mathbf{H}^T\} = \rho^2\,\mathbf{I}$ and $\mathrm{E}\{\mathbf{N}\,\mathbf{N}^T\} = \sigma^2\,\mathbf{I}$ ($\mathbf{I}$ is the $M \times M$ identity matrix). Transmit symbol A, channel coefficients H_i, and noise N_i are statistically independent.

a) Determine the conditional probability density functions $f_{\mathbf{Y}|A}(\mathbf{y}|a = 0)$ and $f_{\mathbf{Y}|A}(\mathbf{y}|a = 1)$.

b) Show that the ML receiver $\hat{A}_{\mathrm{ML}} = \arg\max_{a\in\{0,1\}} f_{\mathbf{Y}|A}(\mathbf{y}|a)$ can be realized by comparing $z \triangleq \frac{1}{M}\|\mathbf{y}\|^2$ with a threshold γ (which does not depend on M). Determine γ.

c) Calculate the conditional mean $\mu_a = \mathrm{E}\{Z|A = a\}$ and the conditional variance $\sigma_a^2 = \mathrm{E}\{(Z - \mu_a)^2|A = a\}$ of $Z = \frac{1}{M}\|\mathbf{Y}\|^2$ for $a = 0$ and $a = 1$. Use your results to show that the error probability of the ML receiver decreases with growing M.

Hint: For a zero-mean Gaussian random variable X, $\mathrm{E}\{X^4\} = 3(\mathrm{E}\{X^2\})^2$.

Problem 2 (20 credits)

Consider a bandpass PAM system with symbol period T_s, transmit pulse $g(t)$ (duration T_s, energy $E_g = 1$), and transmit symbols $A[k]$. The receive pulse in the equivalent baseband is $h(t) = g(t) + \alpha\, g(t - T_s)$ ($|\alpha| < 1$). The additive channel noise has a power spectral density of $N_0/2$. The receiver consists of a sampled matched filter, a noise whitening filter (based on a *causal* factorization), a slicer, and an equalizer.

a) Determine the impulse response, transfer function, and noise power spectral density of the equivalent discrete-time model of the transmission system including the noise whitening filter (but not the equalizer).

b) Calculate the zero-forcing equalizer (placed after the noise whitening filter). Compare the noise variance at the input and the output of the equalizer.

c) Verify that the output of the noise whitening filter is free of precursor intersymbol interference.

d) Determine the decision-feedback equalizer that completely removes postcursor intersymbol interference. *Note:* The noise whitening filter is used as the feedforward filter of the decision-feedback equalizer.

Problem 3 (20 credits)

Consider transmission of a symbol $A \in \{1, 4\}$. At the slicer input, the receiver observes a random variable Q that is corrupted by real-valued multiplicative noise Z, i.e.,

$$Q = A\,Z\,.$$

The noise is Laplace distributed, i.e.,

$$f_Z(z) = e^{-2\,|z-1|}\,.$$

a) Calculate and sketch $f_{Q|A}(q|a)$ for $a = 1$ and $a = 4$.

b) Calculate the probabilities $\mathrm{P}\{Q > \eta_1 \,|\, A = 1\}$ and $\mathrm{P}\{Q < \eta_2 \,|\, A = 4\}$ for $\eta_1 > 1$ and $\eta_2 < 4$.

c) Consider ML detection. Determine the decision rule and sketch the decision intervals. Calculate the conditional symbol error probability $\mathrm{P}\{\mathcal{E} \,|\, A = a\}$ for $a = 1$ and $a = 4$.

d) Consider a detector using the following decision rule:

$$\hat{A} = \begin{cases} 1\,, & \big|Q-1\big| < 0.5 \\ 4\,, & \big|Q-4\big| < 0.5 \\ \text{undecided}, & \text{otherwise.} \end{cases}$$

Sketch the decision intervals. Calculate the conditional symbol error probabilities $\mathrm{P}\{\hat{A} = 1 \,|\, A = 4\}$ and $\mathrm{P}\{\hat{A} = 4 \,|\, A = 1\}$.

Problem 4 (20 credits)

Consider a passband PAM system with an ML sequence detector. In the equivalent discrete-time baseband domain, the channel is described by the folded spectrum

$$S_h(z) = \frac{8z^2 + 65z + 8}{8z}.$$

Furthermore, the channel adds white Gaussian noise.

a) Sketch the function $S_h(e^{j\theta})$ for $\theta \in [-\pi, \pi]$.

b) Find the poles and zeros of $S_h(z)$.

c) Find a minimum phase factorization of $S_h(z)$.

d) Calculate the transfer function and impulse response of the equivalent discrete-time system including the noise whitening filter.

e) Consider the transmission of symbols $A[k] \in \{1, j, -1, -j\}$. Sketch the state transition diagram associated with the equivalent discrete-time system including the noise whitening filter.