Digital Communications 1
Written exam on January 26, 2022

Institute of Telecommunications

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Please note:

- You may use the official lecture notes, a pocket calculator, and a collection of mathematical formulas.
- Personal notes, materials from exercise classes, and pre-calculated problems may not be used.
- Legible writing and a clear layout of your derivations and solutions are absolutely necessary!
- Provide detailed derivations. When using results from the lecture notes, they must be explicitly referenced.

Problem 1 (20 credits)

Consider a passband PAM system with symbol alphabet $\mathcal{A} = \{-2, 2, -3j, 3j\}$. The symbols $A[k]$ are assumed white and uniformly distributed.

a) For transmission, the symbol sequence $A[k]$ is transformed into a sequence $B[k] = A[k] + \alpha A[k-1]$. Calculate the power spectral density $S_B(e^{j\theta})$ of the transformed symbols $B[k]$.

b) The equivalent baseband transmit signal is $S_{\mathrm{LP}}(t) = \sum_{k=-\infty}^{\infty} B[k] g(t - kT_s)$. Calculate the power spectral density $S_{\bar{S}_{\mathrm{LP}}}(j\omega)$ of the stationarized transmit signal $\bar{S}_{\mathrm{LP}}(t)$.

c) The spectrum $S_{\bar{S}_{\mathrm{LP}}}(j\omega)$ is required to be zero at frequency $\omega = \frac{\pi}{T_s}$. Which choice of α satisfies this condition?

d) Assume that the transmit pulse $g(t)$ is a sinc pulse: $g(t) = \mathrm{sinc}(\frac{\pi t}{T_s})$. Sketch the power spectral density of the stationarized transmit signal $S_{\bar{S}_{\mathrm{LP}}}(j\omega)$ for this case, using the α calculated in c).

Problem 2 (20 credits)

White and equally likely symbols $A[k] \in \{-1, 1\}$ are transmitted over a discrete-time channel with linear distortion described by the impulse response $h[k] = \delta[k] - \frac{1}{3}\delta[k-1]$ and additive white Gaussian noise of variance σ_Z^2. At the output of the channel, the sequence $Y[k] = (h*A)[k] + Z[k]$ is received.

a) As a first step, consider direct symbolwise detection of $Y[k]$.

a1) Calculate the conditional probability of a symbol error

$$\mathrm{P}\{\hat{A}[k] \neq A[k] | A[k] = \alpha, A[k-1] = \beta\} \qquad \text{for } \alpha, \beta \in \{-1, 1\}.$$

a2) Calculate the unconditional probability of a symbol error $\mathrm{P}\{\hat{A}[k] \neq A[k]\}$.

b) Now zero-forcing equalization of the sequence $Y[k]$ is considered prior to symbolwise detection.

b1) Find the transfer function and impulse response of the zero-forcing equalizer and calculate the noise variance after equalization.

b2) Calculate the unconditional probability of a symbol error $\mathrm{P}\{\hat{A}[k] \neq A[k]\}$ of this zero-forcing detector and compare it with the probability calculated in part a2).

Problem 3 (20 credits)

Consider transmission of the signals

$$s^{(1)}(t) = A\sin\left(2\pi\,\frac{t}{T}\right)\mathrm{rect}\left(t;\frac{T}{2}\right),$$
$$s^{(2)}(t) = A\cos\left(2\pi\,\frac{t}{T}\right)\mathrm{rect}\left(t;\frac{T}{2}\right),$$

with $T = 2$ ms, over an AWGN channel with noise power spectral density $N_0/2 = 2\cdot 10^{-5}$ W/Hz. The transmit probabilities are given by $p_I(1) = 2/5$.

a) Find an orthonormal basis for the signal set and specify the corresponding basis coefficient vectors $\mathbf{s}^{(1)}$ and $\mathbf{s}^{(2)}$.

b) State the MAP decision rule using inner products between the basis coefficient vectors of the received signal and those of the transmit signals. Sketch a MAP sequence detector implementation that uses these inner products.

c) Since the detection problem corresponds to the binary case, the MAP sequence detector can be implemented using only one basis function. Find a basis function that is suitable for this implementation. State the MAP decision rule using only one basis coefficient of the received signal and of the transmit signals. Sketch the corresponding MAP sequence detector implementation.

d) For $A = 1.01\sqrt{\mathrm{W}}$, calculate the conditional sequence error probabilities $\mathrm{P_{MAP}}\{\mathcal{E}|I=1\}$ and $\mathrm{P_{MAP}}\{\mathcal{E}|I=2\}$ as well as the unconditional sequence error probability $\mathrm{P_{MAP}}\{\mathcal{E}\}$.

e) Find the value of A that reduces the conditional sequence error probability $\mathrm{P_{MAP}}\{\mathcal{E}|I=1\}$ to 10% of the value obtained in d).

Problem 4 (20 credits)

At the transmitter of a communication system, a sequence of binary symbols $a[k] \in \{0, 1\}$ is mapped to a sequence of ternary symbols $b[k] \in \{-1, 0, 1\}$ according to the following rule:

$$\begin{aligned} a[k] = 0 &\implies b[k] = 0, \\ a[k] = 1 &\implies \begin{cases} b[k] = -1 & \text{if } a[k-1] = 0, \\ b[k] = 1 & \text{if } a[k-1] = 1. \end{cases} \end{aligned} \tag{1}$$

The receiver observes the sequence $y[k] = b[k] + n[k]$, where $n[k]$ is white Gaussian noise. The ML detector is used for detecting the symbols $a[k]$.

a) Sketch the state diagram and one stage of the trellis diagram.

b) Use the Viterbi algorithm to identify the path with minimum distance from the zero path $a[k] \equiv 0$. You may assume that the initial state is zero. Determine the corresponding symbol sequence, the distance from the zero path, and the number of symbol errors.

c) Determine the symbol sequence obtained with the ML sequence detector for the received sequence $(y[k]) = \cdots 0, 0, 0, 0.3, -0.8, 0.5, -0.2$ and calculate the corresponding path metric.