# Modulations- und Detektionsverfahren

## Written exam on January 25, 2013

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**Please note:**

- You may use the official lecture notes, a pocket calculator, and a collection of mathematical formulas.
- Personal notes, materials from exercise classes, and pre-calculated problems may not be used.
- Legible writing and a clear layout of your derivations and solutions are absolutely necessary!
- Provide detailed derivations. When using results from the lecture notes, they must be explicitly referenced.

**Problem 1 (20 credits)**

Consider transmission of the signals

$$s^{(1)}(t) = A \sin\left(2\pi \frac{t}{T}\right) \operatorname{rect}\left(t, \frac{T}{2}\right)$$
$$s^{(2)}(t) = A \cos\left(2\pi \frac{t}{T}\right) \operatorname{rect}\left(t, \frac{T}{2}\right),$$

with $T = 2\text{ms}$, over an AWGN channel with noise power spectral density $N_0/2 = 2 \cdot 10^{-5}$ W/Hz. The transmit probabilities are given by $p_I(1) = 2/5$.

**a)** Find an orthonormal basis for the signal set and specify the corresponding basis coefficient vectors $\mathbf{s}^{(1)}$ and $\mathbf{s}^{(2)}$.

**b)** State the MAP decision rule using inner products between the basis coefficient vectors of the received signal and those of the transmit signals. Sketch a MAP sequence detector implementation that uses these inner products.

**c)** Since the detection problem corresponds to the binary case, the MAP sequence detector can be implemented using only one basis function. Find a basis function that is suitable for this implementation. State the MAP decision rule using only one basis coefficient of the received signal and of the transmit signals. Sketch the corresponding MAP sequence detector implementation.

**d)** For $A = 1.01\sqrt{\text{W}}$, calculate the conditional sequence error probabilities $\mathrm{P}_{\text{MAP}}\{\mathcal{E}|I = 1\}$ and $\mathrm{P}_{\text{MAP}}\{\mathcal{E}|I = 2\}$ as well as the unconditional sequence error probability $\mathrm{P}_{\text{MAP}}\{\mathcal{E}\}$.

**e)** Find the value of $A$ that reduces the conditional sequence error probability $\mathrm{P}_{\text{MAP}}\{\mathcal{E}|I = 1\}$ to 10% of the value obtained in d).

**Problem 2 (20 credits)**

Consider passband PAM transmission using the signal constellation

$$\mathcal{A} = \{-1+j, -1-j, 1+j, 1-j, \beta+j, \beta-j\},$$

with $\beta > 1$. The receiver performs symbolwise ML detection. The symbols $a[k] \in \mathcal{A}$ are corrupted by additive noise $z[k] \in \mathbb{C}$, yielding the following signal at the slicer input:

$$q[k] = a[k] + z[k],$$

with real part $q_{\mathrm{R}}[k]$ and imaginary part $q_{\mathrm{I}}[k]$. The real part and the imaginary part of the noise at the slicer input, $z[k] = z_{\mathrm{R}}[k] + j z_{\mathrm{I}}[k]$, are statistically independent zero-mean Gaussian random variables with variances $\sigma_{\mathrm{R}}^2 = 1$ and $\sigma_{\mathrm{I}}^2 = 0.25$, respectively.

**a)** Sketch the signal constellation and the decision regions of the symbolwise ML detector.

**b)** Assume that the symbol $1 + j$ was transmitted. Specify the ML decision region for a correct decision by specifying the intervals within which $q_{\mathrm{R}}[k]$ and $q_{\mathrm{I}}[k]$ must lie.

**c)** Calculate the probability of a correct decision for the case that $1+j$ was transmitted, depending on $\beta$. Find the value of $\beta$ for which this probability is 0.99.

**d)** The prior probabilities of symbols with a positive real part is twice as large as that of symbols with a negative real part. Using the value of $\beta$ calculated in part c), calculate the mean symbol power $P_A$.

**Problem 3 (20 credits)**

In a passband PAM system, the received pulse is given by

$$h(t) = \frac{2t + T_s}{\sqrt{T_s^3}} \operatorname{rect}\left(t + \frac{1}{4}T_s; \frac{T_s}{4}\right) + \frac{1}{2\sqrt{T_s}} \operatorname{rect}\left(t - \frac{1}{2}T_s; \frac{T_s}{2}\right).$$

**a)** Calculate the impulse response $\rho_h[k]$ and the transfer function $S_h(z)$ of the equivalent discrete-time system (including the sampled matched filter).

**b)** Find the zeros and poles of $S_h(z)$.

**c)** Calculate the linear zero forcing equalizer and find its poles.

**d)** Find a minimum-phase factorization of $S_h(z)$.

**e)** Calculate the transfer function and the impulse response of the equivalent discrete-time system including the noise-whitening filter.

## Problem 4 (20 credits)

For two complex signals $x(t)$ and $y(t)$, consider the quantities $\|x\|$, $\|y\|$, and $\langle x, y \rangle$.

**a)** Show how these quantities are changed by the following transformations of *both* signals $x(t)$ and $y(t)$:

**a1)** a frequency shift by $\omega_0$;

**a2)** multiplication by a constant factor $\alpha \in \mathbb{C}$.

**b)** Consider the signals

$$x(t) = |T - t| \operatorname{rect}(t;T), \qquad y(t) = -|T - t - \tau| \operatorname{rect}(t;T) .$$

**b1)** Calculate $\|x\|$, $\|y\|$, and $\langle x, y \rangle$.

**b2)** Sketch $\langle x, y \rangle$ as a function of $\tau$.