Digital Communications 1

Written exam on June 24, 2014

Institute of Telecommunications

Vienna University of Technology

Please note:

- You may use the official lecture notes, a pocket calculator, and a collection of mathematical formulas.
- Personal notes, materials from exercise classes, and pre-calculated problems may not be used.
- Legible writing and a clear layout of your derivations and solutions are absolutely necessary!
- Provide detailed derivations. When using results from the lecture notes, they must be explicitly referenced.

Problem 1 (20 credits)

Consider BPSK signaling on a diversity channel, where the received signal at the output of channel l is

$$r_l(t) = \alpha_l a g(t) + w_l(t), \qquad l = 1, 2.$$

Here, $a \in \{-1, 1\}$ is the transmit symbol, $g(t)$ is the transmit pulse with energy E_g, and the $w_l(t)$ are independent complex WGN processes with power spectral density N_0. The squared channel coefficients $\{\alpha_l^2\}$ are independent and exponentially distributed with $\mathrm{E}\{\alpha_l^2\} = 1$. *Note:* The pdf of an exponentially distributed random variable x with parameter λ is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0, \end{cases}$$

where $\mathrm{E}\{x\} = 1/\lambda$.

The receiver achieves selection diversity by making a decision considering only the stronger channel. More specifically, the receiver first determines the channel with the larger amplitude, i.e., $\hat{l} = \arg\max_{l \in \{1,2\}} \alpha_l$, and then calculates the inner product of the corresponding channel output $r_{\hat{l}}(t)$ with the transmit pulse,

$$c = \langle r_{\hat{l}}, g \rangle.$$

A decision in favor of symbol $+1$ is made if $\mathrm{Re}\{c\} > 0$, and in favor of symbol -1 otherwise.

a) Determine the pdfs of the SNRs $\gamma_l \triangleq \alpha_l^2 E_g / N_0$ at the individual channels $l = 1, 2$. Are the SNRs γ_l independent?

b) Let γ denote the SNR of the stronger channel, i.e., $\gamma = \max\{\gamma_1, \gamma_2\}$. Show that the bit error probability for the system conditioned on γ is given by $\mathrm{P}\{\mathcal{E}_b | \gamma\} = Q(\sqrt{2\gamma})$.

c) Show that the pdf of $\gamma = \max\{\gamma_1, \gamma_2\}$ is given by

$$f(\gamma) = \begin{cases} \frac{2}{\bar{\gamma}}\left(e^{-\gamma/\bar{\gamma}} - e^{-2\gamma/\bar{\gamma}}\right) & \gamma \geq 0 \\ 0 & \gamma < 0. \end{cases}$$

Hint: The cdf of an exponentially distributed random variable x with parameter λ is given by

$$F(x) = 1 - e^{-\lambda x}.$$

d) Show that the bit error probability for the system is given by

$$\mathrm{P}\{\mathcal{E}_b\} = \frac{1}{2}\left[1 - 2\sqrt{\frac{\bar{\gamma}}{1+\bar{\gamma}}} + \sqrt{\frac{\bar{\gamma}}{2+\bar{\gamma}}}\right].$$

Hint:

$$\int_0^\infty Q(\sqrt{x}) \frac{e^{-x/\alpha}}{\alpha} \mathrm{d}x = \frac{1}{2}\left[1 - \sqrt{\frac{\alpha}{2+\alpha}}\right].$$

Problem 2 (20 credits)

Consider a passband PAM system with symbol alphabet $\mathcal{A} = \{-3, 3, -3j, 3j\}$. The symbols $A[k]$ are assumed white and uniformly distributed.

a) For transmission, the symbol sequence $A[k]$ is transformed into a sequence $B[k] = A[k] + \alpha A[k-1]$. Calculate the power spectral density $S_B(e^{j\theta})$ of the transformed symbols $B[k]$.

b) The equivalent baseband transmit signal is $S_{\mathrm{LP}}(t) = \sum_{k=-\infty}^{\infty} B[k] g(t - kT_s)$. Calculate the power spectral density $S_{\bar{S}_{\mathrm{LP}}}(j\omega)$ of the stationarized transmit signal $\bar{S}_{\mathrm{LP}}(t)$.

c) The spectrum $S_{\bar{S}_{\mathrm{LP}}}(j\omega)$ is required to be zero at frequency $\omega = \frac{\pi}{T_s}$. Which choice of α satisfies this condition?

d) Assume that the transmit pulse $g(t)$ is a sinc pulse: $g(t) = \mathrm{sinc}(\frac{\pi t}{T_s})$. Sketch the power spectral density of the stationarized transmit signal $S_{\bar{S}_{\mathrm{LP}}}(j\omega)$ for this case, using the α calculated in c).

Problem 3 (20 credits)

For two complex signals $x(t)$ and $y(t)$, consider the quantities $\|x\|$, $\|y\|$, and $\langle x, y\rangle$.

a) Show how these quantities are changed by the following transformations of *both* signals $x(t)$ and $y(t)$:

a1) a frequency shift by ω_0;

a2) multiplication by a constant factor $\alpha \in \mathbb{C}$.

b) Consider the signals

$$x(t) = \left|\frac{t}{T}\right| \operatorname{rect}(t;T), \qquad y(t) = -\left|\frac{t}{T} - \tau\right| \operatorname{rect}(t;T)\,,$$

with $0 < \tau \leq 1$.

b1) Calculate $\|x\|$, $\|y\|$, and $\langle x, y\rangle$.

b2) Sketch $\langle x, y\rangle$ as a function of τ.

Problem 4 (20 credits)

Asymmetric constellations provide a simple method for unequal error protection, where important bits can be protected more than bits of lesser importance. Consider an asymmetric QPSK constellation as shown in Figure 1, where the mapping of two bits b_1 b_2 into each signal point is also shown. The information bits are equally likely. Communication is performed over an AWGN channel with two-sided power spectral density $\frac{N_0}{2}$. The transmission is ISI-free with gain factor $|p[0]|$.

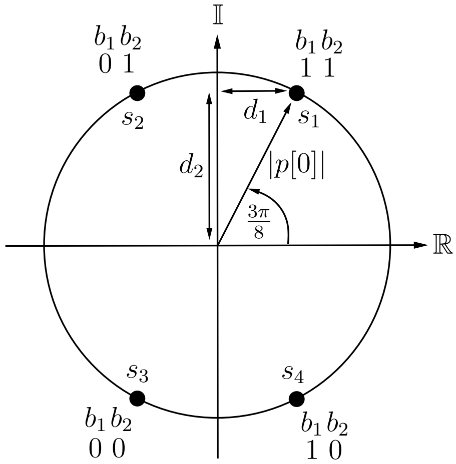

Figure 1: Asymmetric QPSK (as seen at the slicer input).

a) Draw the decision regions of the receiver that minimizes the symbol error probability.

b) Determine the symbol error probability of the receiver in a) as a function of $\frac{|p[0]|}{\sqrt{N_0 E_f}}$, where E_f denotes the energy of the receive filter.

c) Determine the bit error probabilities for bits b_1 and b_2 separately (denoted $\mathrm{P}\{\mathcal{E}_{b_1}\}$ and $\mathrm{P}\{\mathcal{E}_{b_2}\}$). Which bit is more protected and why?

d) Assume that $N_0 = 10^{-6}$ and $E_f = 1$. How large does $|p[0]|$ need to be set to achieve $\mathrm{P}\{\mathcal{E}_{b_1}\} \leq 10^{-3}$? How large is $\mathrm{P}\{\mathcal{E}_{b_2}\}$ at that value of $|p[0]|$?