Digital Communications 1

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Please note:

- You may use the official lecture notes, a pocket calculator, and a collection of mathematical formulas.
- Personal notes, materials from exercise classes, and pre-calculated problems may not be used.
- Legible writing and a clear layout of your derivations and solutions are absolutely necessary!
- Provide detailed derivations. When using results from the lecture notes, they must be explicitly referenced.

Problem 1 (20 credits)

Consider transmission of a single 4PAM symbol S randomly selected from the symbol alphabet $\mathcal{S} \triangleq \{\pm 1, \pm 3\}$. The receiver observes the noise-contaminated "symbol" $Y = S + N$, where $N \in \mathbb{R}$ is Gaussian distributed with variance σ_N^2 and independent of S. An automatic gain control (AGC) circuit is supposed to scale the signal at the receiver so that the noiseless constellation points are indeed at $\{\pm 1, \pm 3\}$. The receiver uses an ML detector whose decision boundaries are set according to this nominal scaling.

a) Consider a faulty AGC that performs a scaling such that the actual noiseless constellation points are at $\{\pm 0.9, \pm 2.7\}$. Normally, AGC circuits try to maintain a constant output power as the input power varies, and can be considered to impose a scale factor on the input that is inversely proportional to the square root of the mean input power. Does the faulty AGC circuit overestimate or underestimate the input power (relative to the above general rule)?

b) Sketch the noiseless constellation points at the output of the faulty AGC and the decision regions of the mismatched ML detector.

c) For the ML detector, calculate the conditional error probabilities $\mathrm{P}_{\mathrm{ML}}\{\mathcal{E}|S\}$ for all $S \in \mathcal{S}$ and the unconditional error probability $\mathrm{P}_{\mathrm{ML}}\{\mathcal{E}\}$.

d) Suppose $p_{\mathcal{S}}(\pm 1) = 1/3$ and $p_{\mathcal{S}}(3) = p_{\mathcal{S}}(-3)$. Calculate the decision regions of the mismatched MAP detector for $\sigma_N^2 = 1/2$.

Problem 2 (20 credits)

Consider replacing the zero-forcing equalizer

$$D_{\mathrm{ZF}}(e^{j\theta}) = \frac{1}{1 - 4\cos\theta}$$

by a decision feedback equalizer. The transmit symbols of the system are taken from the alphabet $\{1, -1\}$ with equal probabilities. The symbol sequence and the equivalent discrete-time noise $Z[k]$ are uncorrelated and both white. The noise is zero-mean, and the noise variance is $\sigma_Z^2 = 1$.

a) Calculate the pulse $p[k]$ of the equivalent discrete-time baseband channel consisting of transmit pulse, physical channel, and matched receive filter.

b) Assume a general feedforward filter $d[k], k \in [-L, L]$. Calculate the overall pulse $p^{(d)}[k]$ at the output of the feedforward filter. How long is this pulse?

c) Consider now the case $L = 1$. What is the minimum length K of the feedback filter such that as much ISI as possible is canceled (under the assumption that all previous symbol decisions were correct)?

d) Calculate the coefficients of the MSE-optimum feedforward filter $\mathbf{d}_{\mathrm{MSE}}$.

Hint: The inverse of a matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

can be calculated as

$$\mathbf{A}^{-1} = \frac{1}{\det\mathbf{A}} \begin{pmatrix} a_{22}a_{33} - a_{23}a_{32} & a_{13}a_{32} - a_{12}a_{33} & a_{12}a_{23} - a_{13}a_{22} \\ a_{23}a_{31} - a_{21}a_{33} & a_{11}a_{33} - a_{13}a_{31} & a_{13}a_{21} - a_{11}a_{23} \\ a_{21}a_{32} - a_{22}a_{31} & a_{12}a_{31} - a_{11}a_{32} & a_{11}a_{22} - a_{12}a_{21} \end{pmatrix},$$

with

$$\det\mathbf{A} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

Problem 3 (20 credits)

For two complex signals $x(t)$ and $y(t)$, consider the quantities $\|x\|$, $\|y\|$, and $\langle x, y\rangle$.

a) Show how these quantities are changed by the following transformations of *both* signals $x(t)$ and $y(t)$:

a1) a time shift by t_0;

a2) multiplication by a constant factor $\alpha \in \mathbb{C}$.

b) Consider the signals

$$x(t) = A\sin(\omega_0 t)\ \mathrm{rect}(t + T/2; T/2), \qquad y(t) = -A\sin(\omega_0 t + \theta)\ \mathrm{rect}(t - T/2; T),$$

with $T = 2\pi/\omega_0$.

b1) Calculate $\|x\|$, $\|y\|$, and $\langle x, y\rangle$.

b2) Sketch $\langle x, y\rangle$ as a function of θ.

Problem 4 (20 credits)

A random variable S from the symbol alphabet $\mathcal{S} \subset \mathbb{C}$ is corrupted by additive noise $N \in \mathbb{C}$ that is statistically independent of S. The real part and imaginary part of the noise, N_{R} and N_{I}, are statistically independent and have Laplacian distributions

$$f_{N_{\mathrm{R}}}(n_{\mathrm{R}}) = \frac{1}{2}\, e^{-|n_{\mathrm{R}}|}, \qquad f_{N_{\mathrm{I}}}(n_{\mathrm{I}}) = \frac{1}{2}\, e^{-|n_{\mathrm{I}}|}.$$

a) What is the ML decision criterion? (*Hint:* Split S and the observed variable Y into their real and imaginary parts: $S = S_{\mathrm{R}} + jS_{\mathrm{I}}$, $Y = Y_{\mathrm{R}} + jY_{\mathrm{I}}$.)

b) Sketch the signal constellation and the ML decision boundaries for the following signal alphabets:

b1) $\mathcal{S} = \{1/2 + j, 1/2 - j\}$;

b2) $\mathcal{S} = \{1/2 + j, -1/2 + j\}$;

b3) $\mathcal{S} = \{1/2 + j, -1/2 - j\}$.

c) Now assume $\mathcal{S} = \{1/2+j, 1/2-j, -1/2+j, -1/2-j\}$. Sketch the signal constellation and the ML decision region for detecting $\hat{S}_{\mathrm{ML}} = 1/2+j$. Assuming that $S = 1/2+j$ was transmitted, calculate the probability of a correct ML detection.

d) Generalizing the result from c), calculate the conditional error probabilities $\mathrm{P}_{\mathrm{ML}}\{\mathcal{E}|S\}$ for all $S \in \mathcal{S}$ and the unconditional error probability $\mathrm{P}_{\mathrm{ML}}\{\mathcal{E}\}$.