Digital Communications 1
Written exam on June 29, 2016

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Please note:

- You may use the official lecture notes, a pocket calculator, and a collection of mathematical formulas.
- Personal notes, materials from exercise classes, and pre-calculated problems may not be used.
- Legible writing and a clear layout of your derivations and solutions are absolutely necessary!
- Provide detailed derivations. When using results from the lecture notes, they must be explicitly referenced.

Problem 1 (20 credits)

Consider passband PAM transmission using the signal constellation

$$\mathcal{A} = \{\pm 1 \pm j, \pm 1 + j\beta\},$$

with $\beta > 1$. The receiver performs symbolwise ML detection. The symbols $a[k] \in \mathcal{A}$ are corrupted by additive white noise $z[k] \in \mathbb{C}$, yielding the following signal at the slicer input:

$$q[k] = a[k] + z[k],$$

with real part $q_{\mathrm{R}}[k]$ and imaginary part $q_{\mathrm{I}}[k]$. The real part and the imaginary part of the noise at the slicer input, $z[k] = z_{\mathrm{R}}[k] + j z_{\mathrm{I}}[k]$, are statistically independent zero-mean Gaussian random variables with variances $\sigma_{\mathrm{R}}^2 = 1$ and $\sigma_{\mathrm{I}}^2 = 0.5$, respectively.

a) Sketch the signal constellation and the decision regions of the symbolwise ML detector.

b) Assume that the symbol $-1 + j$ was transmitted. Specify the ML decision region for a correct decision by specifying the intervals within which $q_{\mathrm{R}}[k]$ and $q_{\mathrm{I}}[k]$ must lie.

c) Calculate the probability of a correct decision for the case that $-1+j$ was transmitted. Find the value of β for which this probability is 0.6.

d) The prior probabilities of symbols with a positive real part are twice as large as those of symbols with a negative real part. Using the value of β calculated in part c), calculate the mean symbol power P_A.

Problem 2 (20 credits)

Consider a passband PAM system with received pulse

$$h(t) = e^{-ct}\mathrm{u}(t), \qquad c > 0,$$

where $\mathrm{u}(t)$ denotes the unit step function. The additive noise is white with power spectral density $S_N(j\omega) = N_0/2$. The receiver consists of a receive filter $f(t)$, a symbol-rate sampler, a zero-forcing (ZF) linear equalizer, and a slicer.

a) Assume that $f(t) = \delta(t)$, i.e., the overall complex baseband pulse is $p(t) = h(t)$.

- **a1)** Calculate the equivalent discrete-time pulse $p[k]$ at the input of the equalizer. (*Hint*: Note that $\mathrm{u}(kT_s) = \mathrm{u}[k]$, i.e., the discrete-time unit step function.)
- **a2)** Determine the variance σ_Z^2 of the equivalent discrete-time noise $Z[k]$.
- **a3)** Calculate the transfer function $D_{\mathrm{ZF}}(e^{j\theta})$ and the impulse response $d_{\mathrm{ZF}}[k]$ of the ZF linear equalizer.
- **a4)** Assume that $S_Z(e^{j\theta}) \equiv 1$. Determine the power spectral density $S_U(e^{j\theta})$ of the noise $U[k]$ at the output of the ZF linear equalizer.

b) Consider the receive filter $f(t) = h^*(-t)$.

- **b1)** Determine the transfer function $F(j\omega)$.
- **b2)** Calculate the equivalent discrete-time pulse $p[k]$ at the input of the equalizer.
- **b3)** Calculate the transfer function $D_{\mathrm{ZF}}(e^{j\theta})$ and the impulse response $d_{\mathrm{ZF}}[k]$ of the ZF linear equalizer.

Problem 3 (20 credits)

A transmitter chooses from four signals represented by the following vectors (within an orthonormal basis expansion):

$$\mathbf{s}^{(1)} = \begin{pmatrix} c \\ 0 \end{pmatrix}, \quad \mathbf{s}^{(2)} = \begin{pmatrix} 0 \\ c \end{pmatrix}, \quad \mathbf{s}^{(3)} = \begin{pmatrix} -c \\ 0 \end{pmatrix}, \quad \mathbf{s}^{(4)} = \begin{pmatrix} 0 \\ -c \end{pmatrix},$$

with $c^2 = 10$ mJ. The transmit probability $p_I(1) = p_I(2)$ is nine times as large as $p_I(3) = p_I(4)$. The signal is transmitted over an AWGN channel with $N_0/2 = 2 \cdot 10^{-4}$ W/Hz.

a) Consider the ML detector. Calculate its decision rule and sketch the decision regions. Calculate the resulting error probability $\mathrm{P}_{\mathrm{ML}}\{\mathcal{E}\}$.

b) Consider the MAP detector. Calculate its decision rule and sketch the decision regions. Calculate the resulting error probability $\mathrm{P}_{\mathrm{MAP}}\{\mathcal{E}\}$.

c) Reconsider the ML detector. By what factor must c be changed to achieve the error probability calculated for the MAP detector in b)?

d) The entire symbol alphabet is shifted by adding a vector $\mathbf{d}$. Calculate the $\mathbf{d}$ that minimizes the transmit signal power **(i)** for the original signal constellation and **(ii)** for the scaled signal constellation calculated in c). How do these shifts affect the error probabilities $\mathrm{P}_{\mathrm{ML}}\{\mathcal{E}\}$ and $\mathrm{P}_{\mathrm{MAP}}\{\mathcal{E}\}$?

Problem 4 (20 credits)

Consider a communication scheme transmitting in each symbol interval of length T_s one of M_a symbols. Symbol $m \in \{1, \ldots, M_a\}$ is represented by the transmit pulse

$$g_m(t) = G \cos\big((2m+1)\omega_0 t\big)\,\mathrm{sinc}(\omega_0 t),$$

where $\mathrm{sinc}(x) = \frac{\sin(x)}{x}$ and $\omega_0 = \frac{\pi}{T_s}$.

a) Calculate and sketch the Fourier transform $G_m(j\omega)$.

b) Show that all shifted transmit pulses $g_m(t - kT_s)$ and $g_n(t - lT_s)$ $(k, l \in \mathbb{Z})$ are orthogonal unless both $k = l$ and $m = n$.

c) Calculate and sketch the transmit bandwidth B_T and the spectral efficiency ν as a function of M_a.