Digital Communications 1
Written exam on January 25, 2018

Institute of Telecommunications

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Please note:

- You may use the official lecture notes, a pocket calculator, and a collection of mathematical formulas.
- Personal notes, materials from exercise classes, and pre-calculated problems may not be used.
- Legible writing and a clear layout of your derivations and solutions are absolutely necessary!
- Provide detailed derivations. When using results from the lecture notes, they must be explicitly referenced.

Problem 1 (20 credits)

Consider passband PAM transmission using the signal constellation

$$\mathcal{A} = \{-1 \pm j, 1 \pm j, \beta \pm j\},$$

with $\beta > 1$. The receiver performs symbolwise ML detection. The symbols $a[k] \in \mathcal{A}$ are corrupted by additive noise $z[k] \in \mathbb{C}$, yielding the following signal at the slicer input:

$$q[k] = a[k] + z[k],$$

with real part $q_{\mathrm{R}}[k]$ and imaginary part $q_{\mathrm{I}}[k]$. The real part and the imaginary part of the noise at the slicer input, $z[k] = z_{\mathrm{R}}[k] + j z_{\mathrm{I}}[k]$, are statistically independent zero-mean Gaussian random variables with variances $\sigma_{\mathrm{R}}^2 = 1$ and $\sigma_{\mathrm{I}}^2 = 0.25$, respectively.

a) Sketch the signal constellation and the decision regions of the symbolwise ML detector.

b) Assume that the symbol $1 - j$ was transmitted. Specify the ML decision region for a correct decision by specifying the intervals within which $q_{\mathrm{R}}[k]$ and $q_{\mathrm{I}}[k]$ must lie.

c) Calculate the probability of a correct decision for the case that $1 - j$ was transmitted. Find the value of β for which this probability is 0.8.

d) The prior probabilities of the four symbols with a positive real part are equal and twice as large as those of the two symbols with a negative real part (which are also equal). Using the value of β determined in part c), calculate the mean symbol power P_A.

Problem 2 (20 credits)

Consider a PAM system with received pulse

$$h(t) = e^{-t/T_s}\mathrm{u}(t) + e^{-(t/T_s - a)}\mathrm{u}(t - aT_s), \qquad \text{where } a > 0.$$

Here, T_s is the symbol rate and $\mathrm{u}(t)$ denotes the unit step function. The noise $N(t)$ is white with power spectral density $N_0/2$. The receiver consists of a bandpass-lowpass transformation, a receive filter $f(t) = h^*(-t)$, a symbol-rate sampler, a linear ZF equalizer, and a slicer.

a) Find a function $\phi(t)$ such that $h(t) = \phi(t) + \phi(t - aT_s)$. Calculate $r_\phi(\tau) = \int_{-\infty}^{\infty} \phi(t)\phi^*(t - \tau)dt$.

b) Calculate the equivalent discrete-time pulse $p[k]$ at the equalizer input.
Hint: Use your result from b).

c) Find the value of a such that $p[0] = 2\,T_s$.

d) Assume that the transmit symbols $A[k]$ are white and equally likely with symbol alphabet $\mathcal{A} = \{1, -1\}$. Determine the value of a that maximizes the SNR at the equalizer input.

Problem 3 (20 credits)

Consider transmission of a symbol $A \in \{-1/2, 1\}$. At the slicer input, the receiver observes a random variable Q that is corrupted by real-valued additive noise Z:

$$Q = A + Z.$$

The noise is exponentially distributed, i.e.,

$$f_Z(z) = \begin{cases} e^{-z} & \text{if } z \geq 0, \\ 0 & \text{if } z < 0. \end{cases}$$

a) Calculate and sketch $f_{Q|A}(q|a)$ for each $a \in \{-1/2, 1\}$.

b) Calculate the probabilities $\mathrm{P}\{Q > \eta_1 | A = -1/2\}$ and $\mathrm{P}\{Q < \eta_2 | A = 1\}$ for $\eta_1 > -1/2$ and $\eta_2 < 1$.

c) Consider ML detection. Determine the decision rule and sketch the decision intervals. Calculate the conditional symbol error probability $\mathrm{P}\{\mathcal{E}|A = a\}$ for each $a \in \{-1/2, 1\}$.

d) Consider a detector using the following decision rule:

$$\hat{A} = \begin{cases} -1/2 & \text{if } -1/2 \leq Q \leq 0, \\ 1 & \text{if } 1 \leq Q \leq 2, \\ \text{undecided} & \text{otherwise.} \end{cases}$$

Sketch the decision intervals. Calculate the conditional symbol error probabilities $\mathrm{P}\{\hat{A} = 1 | A = -1/2\}$ and $\mathrm{P}\{\hat{A} = -1/2 | A = 1\}$.

Problem 4 (20 credits)

Consider an orthogonal multipulse PAM system with M transmit pulses $g_m(t)$ and random transmit symbols $A_m[k]$, $m \in \{1, \dots, M\}$, $k \in \mathbb{Z}$.

a) Derive an expression of the autocorrelation function $R_{\bar{S}}(\tau)$ of the stationarized transmit signal $\bar{S}(t)$ without making any assumptions about the symbol statistics.

b) Simplify $R_{\bar{S}}(\tau)$ from a) for the case that the symbols $A_m[k]$ are zero-mean and uncorrelated with respect to m, i.e., $\mathrm{E}\{A_m[k]A_n^*[l]\} = R_A^{(m)}[k-l]\delta_{m,n}$. Calculate the power spectral density $S_{\bar{S}}(j\omega)$.

c) Further specialize $R_{\bar{S}}(\tau)$ and $S_{\bar{S}}(j\omega)$ from b) for the case where the symbols $A_m[k]$ are uncorrelated with respect to k and have constant (but still m-dependent) variances $\sigma_{A_m}^2$.

d) Assume transmit pulses $g_m(t) = G\cos\big((2m-1)\omega_0 t\big)\,\mathrm{sinc}\big(\pi\frac{t}{T_s}\big)$, $m \in \{1, \dots, M\}$, where $\omega_0 = \pi/T_s$. Calculate and sketch $S_{\bar{S}}(j\omega)$ for zero-mean, uncorrelated and uniformly distributed symbols $A_m[k] \in \{-2, 2\}$.

e) Determine the transmit bandwidth and the spectral efficiency if the transmit pulses from d) are used.