Digital Communications 1

Written exam on October 14, 2014

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Please note:

- You may use the official lecture notes, a pocket calculator, and a collection of mathematical formulas.
- Personal notes, materials from exercise classes, and pre-calculated problems may not be used.
- Legible writing and a clear layout of your derivations and solutions are absolutely necessary!
- Provide detailed derivations. When using results from the lecture notes, they must be explicitly referenced.

Problem 1 (20 credits)

Consider a two-dimensional (2-D) channel $\mathbf{z} = \mathbf{s} + \mathbf{n}$, where $\mathbf{s} = (s_1\, s_2)^{\mathrm{T}} \in \mathcal{S}$ is the transmitted symbol, $\mathbf{z} = (z_1\, z_2)^{\mathrm{T}} \in \mathbb{R}^2$ is the received pseudosymbol, and the noise $\mathbf{n} = (n_1\, n_2)^{\mathrm{T}} \in \mathbb{R}^2$ is real-valued Gaussian with zero mean and covariance matrix $\mathbf{C_n}$. A BPSK-based binary symbol alphabet $\mathcal{S} = \{(1\ 1)^{\mathrm{T}}, (-1\ -1)^{\mathrm{T}}\}$ is used. Both symbols are equally likely.

a) Assume that

$$\mathbf{C}_{\mathbf{n}}^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Draw the decision regions of the optimum receiver that minimizes the symbol error probability. How large is the symbol error probability?

b) Now assume that

$$\mathbf{C}_{\mathbf{n}}^{(2)} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix}.$$

Calculate the symbol error probability of a receiver that uses the decision regions obtained in part a). Is this receiver still optimum? If not, calculate and draw the optimum decision regions.

c) Repeat part b) for

$$\mathbf{C}_{\mathbf{n}}^{(3)} = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}.$$

Hint: The pdf of a 2-D Gaussian random variable $\mathbf{x}$ with mean $\boldsymbol{\mu} = (\mu_1\, \mu_2)^{\mathrm{T}}$ and covariance matrix

$$\mathbf{C} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

is given by

$$f(\mathbf{x}) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)} \left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2} \right] \right).$$

Problem 2 (20 credits)

White and equally likely symbols $A[k] \in \{-1, 1\}$ are transmitted over a discrete-time channel with linear distortion described by the impulse response $h[k] = \delta[k] - \frac{1}{2}\delta[k-1]$ and additive white Gaussian noise of variance σ_z^2. At the output of the channel, the sequence $Y[k] = (h * A)[k] + Z[k]$ is received.

a) As a first step, consider direct symbolwise detection of $Y[k]$.

a1) Calculate the conditional probability of a symbol error

$$\mathrm{P}\{\hat{A}[k] \neq A[k] \,|\, A[k] = \alpha, A[k-1] = \beta\} \qquad \text{for } \alpha, \beta \in \{-1, 1\}.$$

a2) Calculate the unconditional probability of a symbol error $\mathrm{P}\{\hat{A}[k] \neq A[k]\}$.

b) Now zero-forcing equalization of the sequence $Y[k]$ is considered prior to symbolwise detection.

b1) Find the transfer function and impulse response of the zero-forcing equalizer and calculate the noise variance after equalization.

b2) Calculate the unconditional probability of a symbol error $\mathrm{P}\{\hat{A}[k] \neq A[k]\}$ of this zero-forcing detector and compare it with the probability calculated in part a2).

Problem 3 (20 credits)

A real-valued signal $s(t)$ is transmitted over an AWGN channel. The power spectral density of the noise is $N_0/2 = 0.5$. Using an orthonormal basis expansion, the received signal $y(t)$ can be represented by the vector

$$\mathbf{y} = \mathbf{s} + \mathbf{n}\,.$$

The possible transmit vectors $\mathbf{s}^{(i)}$, with $i = 1, \ldots, I$, are the binary vectors of dimension 5, i.e., all possible vectors $\mathbf{s}^{(i)} \in \{0, 1\}^5$.

a) Calculate I, the number of different transmit signals.

b) Consider ML detection of $\mathbf{s}$. Provide an expression of $\hat{\mathbf{s}}_{\text{ML}}$. Calculate $\hat{\mathbf{s}}_{\text{ML}}$ for the received vector

$$\mathbf{y} = \begin{pmatrix} 0.5 \\ -0.3 \\ 0.7 \\ -0.6 \\ 1.2 \end{pmatrix}.$$

c) Consider MAP detection of $\mathbf{s}$, assuming that the prior distribution of $\mathbf{s}$ is given by

$$p(\mathbf{s}) = \begin{cases} \alpha, & \text{if } \sum_{k=1}^{5} s_k \leq 1 \\ 0, & \text{else.} \end{cases}$$

c1) Calculate α.

c2) Calculate the posterior probability of each $\mathbf{s}^{(i)}$ for the $\mathbf{y}$ given in b). What is $\hat{\mathbf{s}}_{\text{MAP}}$ in this case?

Problem 4 (20 credits)

Consider transmission of the signals

$$s^{(i)}(t) = is(t), \quad i \in \{1,2,3\} \quad \text{with } p_I(1) = \frac{p_I(2)}{2} = \frac{1}{5}$$

over an AWGN channel with noise variance $N_0/2$. Here, $s(t)$ is a fixed signal with energy $E_s \neq 0$.

a) Determine an orthonormal basis of the transmit signal space.

b) Specify the vectors corresponding to the signals $s^{(i)}(t)$.

c) Sketch an implementation of the ML detector using inner products of the received signal with the basis functions.

d) Sketch an implementation of the ML detector using inner products of the received signal with the transmit signals.

e) Calculate the conditional sequence error probabilities and the unconditional sequence error probability of the ML detector.