Digital Communications 1
Written exam on June 30, 2017

Institute of Telecommunications

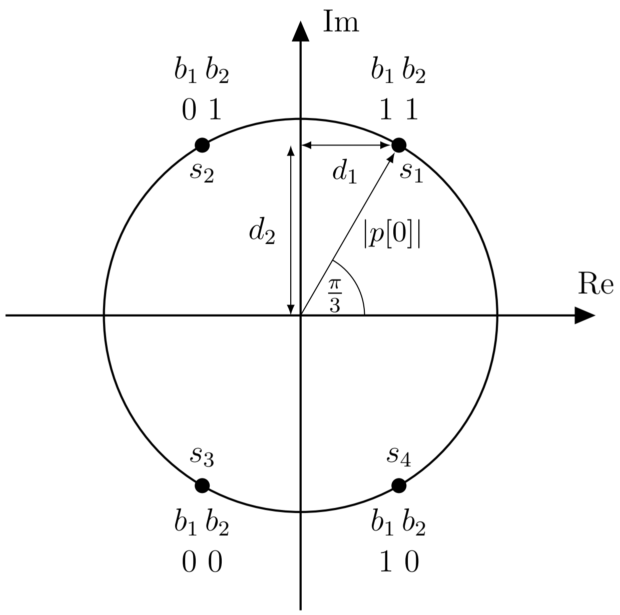
TU Wien

Please note:

- You may use the official lecture notes, a pocket calculator, and a collection of mathematical formulas.
- Personal notes, materials from exercise classes, and pre-calculated problems may not be used.
- Legible writing and a clear layout of your derivations and solutions are absolutely necessary!
- Provide detailed derivations. When using results from the lecture notes, they must be explicitly referenced.

Problem 1 (20 credits)

Asymmetric constellations provide a simple method for unequal error protection, where important bits can be protected more than bits of lesser importance. Consider an asymmetric QPSK constellation as shown in the figure below, where the mapping of two bits b_1, b_2 into each signal point is also shown. The information bits are equally likely. Communication is performed over an AWGN channel with two-sided power spectral density $N_0/2$. The transmission is ISI-free with gain factor $|p[0]|$.



a) Draw the decision regions of the receiver that minimizes the symbol error probability.

b) Determine the symbol error probability of the receiver in a) as a function of $\frac{|p[0]|}{\sqrt{N_0 E_f}}$, where E_f denotes the energy of the receive filter.

c) Determine the bit error probabilities for bits b_1 and b_2 separately (denoted $\mathrm{P}\{\mathcal{E}_{b_1}\}$ and $\mathrm{P}\{\mathcal{E}_{b_2}\}$). Which bit is more protected and why?

d) Assume that $N_0 = 10^{-5}$ and $E_f = 1$. How large does $|p[0]|$ need to be set to achieve $\mathrm{P}\{\mathcal{E}_{b_1}\} \leq 10^{-3}$? How large is $\mathrm{P}\{\mathcal{E}_{b_2}\}$ at that value of $|p[0]|$?

Problem 2 (20 credits)

Consider equalization of a channel with equivalent discrete-time baseband pulse

$$p[k] = \delta[k] - \delta[k-1] - \delta[k+1].$$

The transmit symbols are taken from the alphabet $\{1, -1\}$ with equal probabilities. The symbol sequence and the equivalent discrete-time noise $Z[k]$ are mutually uncorrelated and both white. The noise is zero-mean, and the noise variance is $\sigma_Z^2 = 1$.

a) Calculate $D_{\mathrm{ZF}}(e^{j\theta})$.

b) Next, assume that an MMSE equalizer $d_{\mathrm{MSE}}[k], k \in [-L, L]$ is used. What is a suitable value for L? Why?

c) Sketch the block diagram of an MMSE equalizer using L as determined in b).

d) Calculate $d_{\mathrm{MSE}}[k]$ for $k \in [-L, L]$.

Hint: The inverse of a matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

can be calculated as

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} a_{22}a_{33} - a_{23}a_{32} & a_{13}a_{32} - a_{12}a_{33} & a_{12}a_{23} - a_{13}a_{22} \\ a_{23}a_{31} - a_{21}a_{33} & a_{11}a_{33} - a_{13}a_{31} & a_{13}a_{21} - a_{11}a_{23} \\ a_{21}a_{32} - a_{22}a_{31} & a_{12}a_{31} - a_{11}a_{32} & a_{11}a_{22} - a_{12}a_{21} \end{pmatrix},$$

with

$$\det \mathbf{A} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

Problem 3 (20 credits)

Consider transmission of a symbol A from the alphabet $\mathcal{A} = \{-\frac{d}{4}, \frac{3d}{4}\}$ with $p_A(-\frac{d}{4}) = \frac{3}{4}$. At the slicer input, the receiver observes a random variable Q that is corrupted by real-valued additive noise Z:

$$Q = A + Z.$$

The noise has a triangular distribution given by

$$f_Z(z) = \frac{1}{\epsilon}\left(1 - \frac{|z|}{\epsilon}\right) \mathrm{rect}(z; \epsilon)$$

with parameter $\epsilon > 0$.

a) Sketch $f_Z(z)$.

b) For the following assume $\epsilon = d/2$.

b1) Sketch the weighted distributions $f_{Q|A}(q|a)p_A(a)$ for $a \in \mathcal{A}$.

b2) Determine the MAP decision rule and sketch the decision intervals. Calculate the symbol error probability of the MAP detector.

c) Repeat part b) under the assumption $\epsilon = d$.

Problem 4 (20 credits)

Consider a variant of FSK using the transmit pulses

$$g_m(t) = G\cos\big((2m+1)\omega_0 t\big)\,\mathrm{sinc}(\omega_0 t), \qquad m = 1,\dots,M_a,$$

where $\mathrm{sinc}(x) = \frac{\sin(x)}{x}$ and $\omega_0 = \frac{\pi}{T_s}$.

a) Calculate and sketch the Fourier transform $G_m(j\omega)$.

b) Show that all shifted transmit pulses $g_m(t-kT_s)$ and $g_n(t-lT_s)$ ($k,l \in \mathbb{Z}$) are orthogonal unless both $k=l$ and $n=m$.

c) Calculate and sketch the transmit bandwidth B_T and the spectral efficiency ν as a function of M_a.