Digital Communications 1

Written exam on October 14, 2013

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Please note:

- You may use the official lecture notes, a pocket calculator, and a collection of mathematical formulas.
- Personal notes, materials from exercise classes, and pre-calculated problems may not be used.
- Legible writing and a clear layout of your derivations and solutions are absolutely necessary!
- Provide detailed derivations. When using results from the lecture notes, they must be explicitly referenced.

Problem 1 (20 credits)

Consider equalization of a channel with the equivalent discrete-time baseband pulse

$$p[k] = \delta[k] - \delta[k-1] - \delta[k+1].$$

The transmit symbols are taken from the alphabet $\{1, -1\}$ with equal probabilities. The symbol sequence and the equivalent discrete-time noise $Z[k]$ are uncorrelated and both white. The noise is zero-mean, and the noise variance is $\sigma_Z^2 = 1$.

a) Calculate $D_{\mathrm{ZF}}(e^{j\theta})$.

b) Now, assume that an MMSE equalizer $d_{\mathrm{MSE}}[k], k \in [-L, L]$ is used. What is a suitable value for L? Why?

c) Sketch the block diagram of an MMSE equalizer using L as determined in b).

b) Calculate $d_{\mathrm{MSE}}[k]$ for $k \in [-L, L]$.

Hint: The inverse of a matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

can be calculated as

$$\mathbf{A}^{-1} = \frac{1}{\det\mathbf{A}} \begin{pmatrix} a_{22}a_{33} - a_{23}a_{32} & a_{13}a_{32} - a_{12}a_{33} & a_{12}a_{23} - a_{13}a_{22} \\ a_{23}a_{31} - a_{21}a_{33} & a_{11}a_{33} - a_{13}a_{31} & a_{13}a_{21} - a_{11}a_{23} \\ a_{21}a_{32} - a_{22}a_{31} & a_{12}a_{31} - a_{11}a_{32} & a_{11}a_{22} - a_{12}a_{21} \end{pmatrix},$$

with

$$\det\mathbf{A} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

Problem 2 (20 credits)

Consider a passband PAM system with an ML sequence detector. In the equivalent discrete-time baseband domain, the channel is described by the folded spectrum

$$S_h(z) = \frac{6z^2 + 37z + 6}{6z}.$$

Furthermore, the channel adds white Gaussian noise.

a) Find the poles and zeros of $S_h(z)$.

b) Find a minimum phase factorization of $S_h(z)$.

c) Calculate the transfer function and impulse response of the equivalent discrete-time system including the noise whitening filter.

d) Consider the transmission of statistically independent symbols $A[k] \in \{1, j, -1, -j\}$ with equal probabilities. Assume that the Viterbi algorithm is used at the output of the noise whitening filter. Sketch the state transition diagram.

e) Assume that the receive filter is the matched filter, i.e., $f(t) = h^*(-t)$. Calulate the frequency response $D_{\mathrm{ZF}}(e^{j\theta})$ of the corresponding zero forcing equalizer.

Problem 3 (20 credits)

Consider the transmission of a symbol $A \in \{0, 2\}$ over a single-input, multiple-output (SIMO) system using one transmit antenna and two receive antennas. The input-output relation of the SIMO system is

$$\mathbf{y} = A\mathbf{h} + \mathbf{n},$$

where $\mathbf{y} = (Y_1\ Y_2)^{\mathrm{T}}$ denotes the received vector, $\mathbf{h} = (0.5\ 1)^{\mathrm{T}}$ describes the SIMO channel, and $\mathbf{n} = (N_1\ N_2)^{\mathrm{T}}$ is the additive noise. All variables are real-valued. The noise is jointly Gaussian distributed with mean $\mathrm{E}\{\mathbf{n}\} = \mathbf{0}$ and correlation matrix $\mathbf{R}_{\mathbf{n}} = \mathrm{E}\{\mathbf{n}\,\mathbf{n}^T\} = \sigma_n^2\,\mathbf{I}$.

a) For each $a \in \{0, 2\}$, calculate and sketch $f(\mathbf{y}|a)$ in the $(y_1,\, y_2)$ plane.

b) Consider ML detection of A. Calculate the ML decision rule and sketch the corresponding decision regions.

c) Consider MAP detection of A, assuming that the prior distribution of A is given by

$$p_A(a) = \begin{cases} 3/4, & a = 0 \\ 1/4, & a = 2\,. \end{cases}$$

Calculate the posterior probability of each $A \in \{0, 2\}$ for $y_1 = 1$, $y_2 = -0.5$, and $\sigma_n^2 = 1$. What is $\hat{A}_{\mathrm{MAP}}$ in this case?

d) Consider an alternative receiver using a "matched filter" whose output is

$$Z = \mathbf{h}^T\mathbf{y}.$$

d1) Express Z as a function of A, N_1, and N_2. Calculate $f_{Z|A}(z|a)$.

d2) Find the decision rule of the ML detector using Z.

Problem 4 (20 credits)

Consider the following pulse set:

$$
\begin{aligned}
g_1(t) &= \operatorname{sinc}\left(\frac{\pi}{2}\frac{t}{T_s}\right)\cos\left(3\pi\frac{t}{T_s}\right)\\
g_2(t) &= \operatorname{sinc}\left(\pi\frac{t}{T_s}\right)\cos\left(5\pi\frac{t}{T_s}\right)\\
g_3(t) &= \operatorname{sinc}\left(\frac{\pi}{2}\frac{t}{T_s}\right)\cos\left(\frac{5\pi}{2}\frac{t}{T_s}\right).
\end{aligned}
$$

a) Which of the pulses satisfy the Nyquist criterion? (A derivation is required.)

b) Show that the set of all pulses $g_i(t)$ that satisfy the Nyquist criterion also satisfies the generalized Nyquist criterion.

c) Using the pulses from b), consider orthogonal multipulse PAM with symbol alphabet $\mathcal{A} = \{-1, 1\}$. The symbols $A_m[k]$ are uniformly distributed and uncorrelated, i.e., $\mathrm{E}\{A_m[k]A_n^*[l]\} = \delta_{k,l}\delta_{m,n}$. Sketch the power spectral density $S_{\bar{S}}(j\omega)$ of the stationarized transmit signal $\bar{S}(t)$.