Digital Communications 1
Written exam on October 18, 2018

Institute of Telecommunications

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Please note:

- You may use the official lecture notes, a pocket calculator, and a collection of mathematical formulas.
- Personal notes, materials from exercise classes, and pre-calculated problems may not be used.
- Legible writing and a clear layout of your derivations and solutions are absolutely necessary!
- Provide detailed derivations. When using results from the lecture notes, they must be explicitly referenced.

Problem 1 (20 credits)

Consider a passband PAM system with symbol alphabet $\mathcal{A} = \{-3, 3, -3j, 3j\}$. The symbols $A[k]$ are assumed white and uniformly distributed.

a) For transmission, the symbol sequence $A[k]$ is transformed into a sequence $B[k] = A[k] + \alpha A[k-1]$. Calculate the power spectral density $S_B(e^{j\theta})$ of the transformed symbols $B[k]$.

b) The equivalent baseband transmit signal is $S_{\mathrm{LP}}(t) = \sum_{k=-\infty}^{\infty} B[k]\, g(t - kT_s)$. Calculate the power spectral density $S_{\bar{S}_{\mathrm{LP}}}(j\omega)$ of the stationarized transmit signal $\bar{S}_{\mathrm{LP}}(t)$.

c) The power spectral density $S_{\bar{S}_{\mathrm{LP}}}(j\omega)$ is required to be zero at frequency $\omega = \frac{\pi}{T_s}$. Which choice of α satisfies this condition?

d) Assume that the transmit pulse $g(t)$ is a sinc pulse: $g(t) = \mathrm{sinc}(\frac{\pi t}{T_s})$. Sketch the power spectral density of the stationarized transmit signal $S_{\bar{S}_{\mathrm{LP}}}(j\omega)$ for this case, using the value of α calculated in c).

Problem 2 (20 credits)

Consider passband PAM transmission of symbols $A[k]$ that are taken from the alphabet $\{2, -2\}$ with equal probabilities. The symbol sequence $A[k]$ and the equivalent discrete-time noise $Z[k]$ are independent and both white. The noise is zero-mean and complex Gaussian with variance $\sigma_Z^2 = 1/4$. The receiver consists of a receive filter and a slicer with decision threshold 0. The equivalent discrete-time baseband pulse is

$$p[k] = \delta[k] - \frac{1}{2}\delta[k-1].$$

a) Specify the signal $Q[k]$ at the slicer input and indicate the components constituting 1) the desired signal, 2) ISI, and 3) additive noise.

b) Find the likelihood function $f_{Q[k]\,|\,A[k],A[k-1]}(q[k]\,|\,a[k], a[k-1])$ and calculate the conditional error probabilities $\mathrm{P}\{\hat{A}[k] \neq A[k]\,|\,A[k], A[k-1]\}$ for all four cases $(A[k], A[k-1]) \in \{2, -2\}^2$. Calculate the unconditional error probability $\mathrm{P}\big\{\hat{A}[k] \neq A[k]\big\}$.

c) Calculate the mean powers of the three signal components from a).

Now, consider a receiver that uses a decision feedback equalizer. The coefficients of the feedforward filter and of the feedback filter are $\mathbf{d}_{\mathrm{MSE}} = (-1 \;\; 34 \;\; 0)^T$ and $\mathbf{v}_{\mathrm{MSE}} = (-1 \;\; 0)^T$, respectively.

d) Specify the signal at the slicer input and indicate the components constituting 1) the desired signal, 2) ISI, and 3) additive noise (assuming that all previous symbol decisions were correct).

e) Calculate the mean powers of the three signal components from d).

Problem 3 (20 credits)

Consider a two-dimensional (2-D) channel $\mathbf{z} = \mathbf{s} + \mathbf{n}$, where $\mathbf{s} = (s_1 \;\; s_2)^T \in \mathcal{S}$ is the transmitted symbol, $\mathbf{z} = (z_1 \; z_2)^T \in \mathbb{R}^2$ is the received pseudosymbol, and the noise $\mathbf{n} = (n_1 \; n_2)^T \in \mathbb{R}^2$ is real-valued Gaussian with zero mean and covariance matrix $\mathbf{C_n}$. A BPSK-based binary symbol alphabet $\mathcal{S} = \{(1 \;\; 1)^T, (-1 \;\; -1)^T\}$ is used. Both symbols are equally likely.

a) Assume that

$$\mathbf{C_n} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Draw the decision regions of the optimum receiver that minimizes the symbol error probability. How large is the symbol error probability?

b) Now assume that

$$\mathbf{C_n} = \begin{pmatrix} 1/2 & 0 \\ 0 & 3/2 \end{pmatrix}.$$

Calculate the symbol error probability of a receiver that uses the decision regions obtained in part a). Is this receiver still optimum? If not, calculate and draw the optimum decision regions.

c) Repeat part b) for

$$\mathbf{C_n} = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}.$$

Hint: The pdf of a 2-D Gaussian random variable $\mathbf{x}$ *with mean* $\boldsymbol{\mu} = (\mu_1 \; \mu_2)^T$ *and covariance matrix*

$$\mathbf{C} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

is given by

$$f(\mathbf{x}) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right]\right).$$

Problem 4 (20 credits)

Consider a variant of FSK using the transmit pulses

$$g_m(t) = G\cos\big((2m+1)\omega_0 t\big)\,\mathrm{sinc}(\omega_0 t), \qquad m = 1, \ldots, M_a$$

where $\mathrm{sinc}(x) = \frac{\sin(x)}{x}$ and $\omega_0 = \frac{\pi}{T_s}$.

a) Calculate and sketch the Fourier transform $G_m(j\omega)$.

b) Show that all shifted transmit pulses $g_m(t-kT_s)$ and $g_n(t-lT_s)$ $(k, l \in \mathbb{Z})$ are orthogonal unless both $k = l$ and $n = m$.

c) Calculate and sketch the transmit bandwidth B_T and the spectral efficiency ν as a function of M_a.