Digital Communications 1
Written exam on October 17, 2019

Institute of Telecommunications

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Please note:

- You may use the official lecture notes, a pocket calculator, and a collection of mathematical formulas.
- Personal notes, materials from exercise classes, and pre-calculated problems may not be used.
- Legible writing and a clear layout of your derivations and solutions are absolutely necessary!
- Provide detailed derivations. When using results from the lecture notes, they must be explicitly referenced.

Problem 1 (20 credits)

Consider three base stations B_i, $i = 1, 2, 3$, transmitting an identical symbol sequence $a[k]$ to a single receiver. The base stations use passband PAM transmission with symbol alphabet $\mathcal{A} = \{-1, 1\}$ and equally likely transmit symbols. The Fourier transform of the transmit pulses is given by $G_i(j\omega) = \sqrt{G_i R(j\omega)}$, where $G_1 = 0.002$, $G_2 = 0.01$, $G_3 = 0.005$ and $R(j\omega)$ is the Fourier transform of a raised-cosine pulse. The three channels between the base stations and the receiver are AWGN channels with identical power spectral density $S_N(j\omega) = 0.5 \cdot 10^{-3}$W/Hz, no inter-channel interference, and statistically independent noise processes. The receiver uses a bank of three sampled matched filters (i.e., one filter for each base station signal) whose outputs are the pseudosymbol sequences $q_i[k]$, $i = 1, 2, 3$, followed by a detector.

a) Specify and sketch the equivalent discrete-time baseband PAM system with the single input $a[k]$ and the three outputs $q_i[k]$, $i = 1, 2, 3$. In particular, calculate the total pulses $p_i[k]$ for channels $i = 1, 2, 3$. Are the individual transmissions ISI-free?

We will now consider two different detector designs.

b) *Selection combining:* The detector uses only a single pseudosymbol sequence $q_i[k]$ to decide upon the transmitted symbol sequence. Determine the ML decision rules and calculate the symbol error probabilities for the three choices $i = 1, 2, 3$. Which choice results in the smallest symbol error probability?

c) *Optimum combining:* The detector processes all three pseudosymbol sequences to make a decision about the transmitted symbol sequence. Derive the ML detector for this case. *(Hint:* use the joint channel pdf $f(q_1[k], q_2[k], q_3[k]|a[k])$.*)* Sketch an implementation that uses only one symbol-wise slicer.

d) Calculate the symbol error probability of the optimal receiver from c) and compare it with the symbol error probability of the receiver from b). Interpret the results.

Problem 2 (20 credits)

Consider replacing the zero-forcing equalizer

$$D_{\text{ZF}}(e^{j\theta}) = \frac{1}{1 - 2\cos(\theta)}$$

by a decision feedback equalizer. The transmit symbols of the system are taken from the alphabet $\{1, -1\}$ with equal probabilities. The symbol sequence and the equivalent discrete-time noise $Z[k]$ are uncorrelated and both white. The noise is zero-mean, and the noise variance is $\sigma_Z^2 = 1$.

a) Calculate the pulse $p[k]$ of the equivalent discrete-time baseband channel consisting of transmit pulse, physical channel, and matched receive filter.

b) Assume a general feedforward filter $d[k], k \in [-L, L]$. Calculate the overall pulse $p^{(d)}[k]$ at the output of the feedforward filter. How long is this pulse?

c) Consider now the case $L = 1$. What is the minimum length K of the feedback filter such that as much ISI as possible is canceled (under the assumption that all previous symbol decisions were correct)?

d) Calculate the coefficients of the MSE-optimum feedforward filter $\mathbf{d}_{\text{MSE}}$.

Hint: The inverse of a matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

can be calculated as

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} a_{22}a_{33} - a_{23}a_{32} & a_{13}a_{32} - a_{12}a_{33} & a_{12}a_{23} - a_{13}a_{22} \\ a_{23}a_{31} - a_{21}a_{33} & a_{11}a_{33} - a_{13}a_{31} & a_{13}a_{21} - a_{11}a_{23} \\ a_{21}a_{32} - a_{22}a_{31} & a_{12}a_{31} - a_{11}a_{32} & a_{11}a_{22} - a_{12}a_{21} \end{pmatrix},$$

with

$$\det \mathbf{A} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

Problem 3 (20 credits)

Consider transmission of a symbol $A \in \{-1/2, 1\}$. At the slicer input, the receiver observes a random variable Q that is corrupted by real-valued additive noise Z:

$$Q = A + Z.$$

The noise is exponentially distributed, i.e.,

$$f_Z(z) = \begin{cases} e^{-z} & \text{if } z \geq 0, \\ 0 & \text{if } z < 0. \end{cases}$$

a) For each $a \in \{-1/2, 1\}$, calculate and sketch $f_{Q|A}(q|a)$.

b) Calculate the probabilities $\mathrm{P}\{Q > \eta_1 | A = -1/2\}$ and $\mathrm{P}\{Q < \eta_2 | A = 1\}$ for $\eta_1 > -1/2$ and $\eta_2 < 1$.

c) Consider ML detection. Determine the decision rule and sketch the decision intervals. Calculate the conditional symbol error probability $\mathrm{P}\{\mathcal{E}|A = a\}$ for each $a \in \{-1/2, 1\}$.

d) Consider a detector using the following decision rule:

$$\hat{A} = \begin{cases} -1/2 & \text{if } -1/2 \leq Q \leq 0, \\ 1 & \text{if } 1 \leq Q \leq 2, \\ \text{undecided} & \text{otherwise.} \end{cases}$$

Sketch the decision intervals. Calculate the conditional symbol error probabilities $\mathrm{P}\{\hat{A} = 1 | A = -1/2\}$ and $\mathrm{P}\{\hat{A} = -1/2 | A = 1\}$.

Problem 4 (20 credits)

Consider a variant of FSK using the transmit pulses

$$g_m(t) = G\cos\big((2m+1)\omega_0 t\big)\,\mathrm{sinc}(\omega_0 t), \qquad m = 1, \ldots, M_a$$

where $\mathrm{sinc}(x) = \frac{\sin(x)}{x}$ and $\omega_0 = \frac{\pi}{T_s}$.

a) Calculate and sketch the Fourier transform $G_m(j\omega)$.

b) Show that all shifted transmit pulses $g_m(t - kT_s)$ and $g_n(t - lT_s)$ $(k, l \in \mathbb{Z})$ are orthogonal unless both $k = l$ and $n = m$.

c) Calculate and sketch the transmit bandwidth B_T and the spectral efficiency ν as a function of M_a.