# Modulations- und Detektionsverfahren

## Written exam on November 30, 2011

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**Please note:**

- You may use the official lecture notes, a pocket calculator, and a collection of mathematical formulas.
- Personal notes, materials from exercise classes, and pre-calculated problems may not be used.
- Legible writing and a clear layout of your derivations and solutions are absolutely necessary!
- Provide detailed derivations. When using results from the lecture notes, they must be explicitly referenced.

## Problem 1 (20 credits)

A real-valued signal $s(t)$ is transmitted over an AWGN channel. The power spectral density of the noise is $N_0/2 = 0.5$. Using an orthonormal basis expansion, the received signal $y(t)$ can be represented by the vector

$$\mathbf{y} = \mathbf{s} + \mathbf{n}\,.$$

The possible transmit vectors $\mathbf{s}^{(i)}$, with $i = 1, \ldots, I$, are the binary vectors of dimension 5, i.e., all possible vectors $\mathbf{s}^{(i)} \in \{0,1\}^5$.

**a)** Calculate $I$, the number of different transmit signals.

**b)** Consider ML detection of $\mathbf{s}$. Provide an expression of $\hat{\mathbf{s}}_{\mathrm{ML}}$. Calculate $\hat{\mathbf{s}}_{\mathrm{ML}}$ for the received vector

$$\mathbf{y} = \begin{pmatrix} -0.5 \\ 0.2 \\ 0.8 \\ 1.3 \\ 0.7 \end{pmatrix}.$$

*Hint:* Use the fact that $\mathbf{s} \in \{0,1\}^5 \iff s_1 \in \{0,1\}, \ldots, s_5 \in \{0,1\}$.

**c)** Consider MAP detection of $\mathbf{s}$, assuming that the prior distribution of $\mathbf{s}$ is given by

$$p(\mathbf{s}) = \begin{cases} \alpha, & \text{if } \sum_{k=1}^{5} s_k \le 1 \\ 0, & \text{else.} \end{cases}$$

**c1)** Calculate $\alpha$.

**c2)** Calculate the posterior probability of each $\mathbf{s}^{(i)}$ for the $\mathbf{y}$ given in b). What is $\hat{\mathbf{s}}_{\mathrm{MAP}}$ in this case?

**Problem 2 (20 credits)**

Consider the following pulse set:

$$\begin{aligned}
g_1(t) &= \operatorname{sinc}\left(\pi \frac{t}{T_s}\right) \cos\left(2\pi \frac{t}{T_s}\right) \\
g_2(t) &= \operatorname{sinc}\left(\frac{\pi}{2} \frac{t}{T_s}\right) \cos\left(\frac{7\pi}{2} \frac{t}{T_s}\right) \\
g_3(t) &= \operatorname{sinc}\left(\frac{\pi}{3} \frac{t}{T_s}\right) \cos\left(2\pi \frac{t}{T_s}\right).
\end{aligned}$$

**a)** Sketch the pulses in the frequency domain. Which of the pulses satisfy the Nyquist criterion? (A derivation is required.)

**b)** Show that those pulses $g_i(t)$ that satisfy the Nyquist criterion also satisfy the generalized Nyquist criterion.

**c)** Using the same pulses as in b), consider orthogonal multipulse PAM with symbol alphabet $\mathcal{A} = \{-1, 1\}$. The symbols $A_m[k]$ are uniformly distributed and uncorrelated, i.e., $\mathrm{E}\{A_m[k]A_n^*[l]\} = \delta_{k,l}\delta_{m,n}$. Calculate and sketch the power spectral density $S_{\bar{S}}(j\omega)$ of the stationarized transmit signal $\bar{S}(t)$.

**Problem 3 (20 credits)**

A random variable $S$ from a symbol alphabet $\mathcal{S} \subset \mathbb{C}$ is corrupted by additive noise $N \in \mathbb{C}$ that is statistically independent of $S$. The real part and imaginary part of the noise, $N_\mathrm{R}$ and $N_\mathrm{I}$, are statistically independent and have Laplacian distributions

$$f_{N_\mathrm{R}}(n_\mathrm{R}) = 2\, e^{-4|n_\mathrm{R}|} \qquad f_{N_\mathrm{I}}(n_\mathrm{I}) = 2\, e^{-4|n_\mathrm{I}|}.$$

**a)** What is the ML decision criterion? (*Hint:* Split $S$ and the observed variable $Y$ into their real and imaginary parts: $S = S_\mathrm{R} + jS_\mathrm{I}$, $Y = Y_\mathrm{R} + jY_\mathrm{I}$.)

**b)** Sketch the signal constellation and the ML decision boundaries for the following symbol alphabets:

**b1)** $\mathcal{S} = \{1, j\}$;

**b2)** $\mathcal{S} = \{1, -j\}$;

**b3)** $\mathcal{S} = \{1, -1\}$.

**c)** Now assume $\mathcal{S} = \{1, j, -1, -j\}$. Sketch the signal constellation and the ML decision region for detecting $\hat{S}_\mathrm{ML} = 1$. Assuming that $S = 1$ was transmitted, calculate the probability of a correct ML detection.

**d)** Using the result from c), calculate the conditional error probabilities $\mathrm{P}_\mathrm{ML}\{\mathcal{E}|S\}$ for all $S \in \mathcal{S}$ and the unconditional error probability $\mathrm{P}_\mathrm{ML}\{\mathcal{E}\}$.

**Problem 4 (20 credits)**

Statistically independent symbols $a[k]$ from the alphabet $\mathcal{A} = \{-1, 1\}$ with equal probabilities are transmitted over a discrete-time channel with impulse response

$$h[k] = \delta[k] - \frac{2}{3}\delta[k-1]$$

and additive white Gaussian noise $z[k]$ of variance $\sigma_z^2$. The received signal is $y[k] = (h * a)[k] + z[k]$.

**a)** Consider symbolwise detection according to

$$\hat{a}[k] = \min_{a \in \mathcal{A}} \left| y[k] - a \right| .$$

**a1)** Calculate the conditional symbol error probabilities $\mathrm{P}\{\hat{a}[k] \neq a[k] \,|\, a[k] = \alpha, a[k-1] = \beta\}$ for $\alpha, \beta \in \{-1, 1\}$.

**a2)** Calculate the symbol error probability $\mathrm{P}\{\hat{a}[k] \neq a[k]\}$.

**b)** Now consider a different detector which consists of a *zero forcing equalizer* followed by a symbolwise detector.

**b1)** Calculate the impulse response of the zero forcing equalizer in the frequency and time domains and calculate the variance of the filtered noise at the equalizer output.

**b2)** Calculate the symbol error probability $\mathrm{P}\{\hat{a}[k] \neq a[k]\}$ of the zero forcing detector and compare it to the symbol error probability calculated in a2).