Digital Communications 1
Written exam on May 3, 2016

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Please note:

- You may use the official lecture notes, a pocket calculator, and a collection of mathematical formulas.
- Personal notes, materials from exercise classes, and pre-calculated problems may not be used.
- Legible writing and a clear layout of your derivations and solutions are absolutely necessary!
- Provide detailed derivations. When using results from the lecture notes, they must be explicitly referenced.

Problem 1 (20 credits)

A stationary and white bit sequence (bit rate $R_b = 64\,\mathrm{kbit/s}$) is to be transmitted over a bandlimited AWGN channel (power spectral density $S_N(j\omega) = N_0/2 = 9\cdot 10^{-7}\,\mathrm{W/Hz}$, bandwidth $B_\mathrm{T} = 24\,\mathrm{kHz}$). The modulation scheme is passband PAM using an M_a-ary QAM constellation. The Fourier transform of the transmit pulse is given by $G(j\omega) = \sqrt{R(j\omega)}$, where $R(j\omega)$ is the Fourier transform of a raised-cosine pulse with roll-off factor $\alpha = 0.5$.

a) Determine the size of the symbol alphabet M_a and sketch the corresponding symbol constellation.

b) Design a receive filter for ISI-free transmission.

c) Sketch the block diagram of the ML receiver.

For the following, assume that symbols with positive real part are twice as likely than symbols with negative real part.

d) Calculate the minimum symbol distance d_a for a symbol error probability of $\mathrm{P}\{\mathcal{E}_s\} = 10^{-7}$. Use for this the approximation $\mathrm{P}\{\mathcal{E}_s\} \approx \overline{\mathcal{N}}\,\mathcal{Q}\left(\frac{d_\mathrm{min}}{\sqrt{2}\sigma_Z}\right)$.

e) How does the symbol constellation need to be shifted to minimize the mean transmit power $P_{\bar{S}}$? Sketch the shifted constellation. How large is the achieved reduction of $P_{\bar{S}}$?

Problem 2 (20 credits)

Consider equalization of a channel with the equivalent discrete-time baseband pulse

$$p[k] = \delta[k] + \delta[k-1] + \delta[k+1].$$

The transmit symbols are taken from the alphabet $\{1, -1\}$ with equal probabilities. The symbol sequence and the equivalent discrete-time noise $Z[k]$ are uncorrelated and both white. The noise is zero-mean, and the noise variance is $\sigma_Z^2 = 1$.

a) Calculate $D_{\mathrm{ZF}}(e^{j\theta})$.

b) Now, assume that an MSE equalizer $d_{\mathrm{MSE}}[k], k \in [-L, L]$ is used. What is a suitable value for L? Why?

c) Sketch the block diagram of an MSE equalizer using L as determined in b).

d) Calculate $d_{\mathrm{MSE}}[k]$ for $k \in [-L, L]$.

Hint: The inverse of a matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

can be calculated as

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} a_{22}a_{33} - a_{23}a_{32} & a_{13}a_{32} - a_{12}a_{33} & a_{12}a_{23} - a_{13}a_{22} \\ a_{23}a_{31} - a_{21}a_{33} & a_{11}a_{33} - a_{13}a_{31} & a_{13}a_{21} - a_{11}a_{23} \\ a_{21}a_{32} - a_{22}a_{31} & a_{12}a_{31} - a_{11}a_{32} & a_{11}a_{22} - a_{12}a_{21} \end{pmatrix},$$

with

$$\det \mathbf{A} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

Problem 3 (20 credits)

A BPSK symbol $S \in \{-1, 1\}$ with prior probability $p_S(1) = p_1$ is corrupted by additive noise $N \in \mathbb{R}$ that is statistically independent of S and distributed as follows:

$$f_N(n) = \begin{cases} \alpha(2+n) & \text{if } -2 \le n \le -1, \\ \alpha & \text{if } -1 \le n \le 0, \\ \alpha(1-n/2) & \text{if } \quad 0 \le n \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the received random variable is $Y = S + N$.

a) Sketch $f_N(n)$ and calculate α.

b) Sketch $f_{Y|S}(y|s)$ for $s \in \{-1, 1\}$. Calculate the ML decision threshold η_{ML}.

c) Calculate the conditional error probabilities $\mathrm{P_{ML}}\{\mathcal{E}|S = s\}$ for $s \in \{-1, 1\}$ and the unconditional error probability $\mathrm{P_{ML}}\{\mathcal{E}\}$ of the ML detector.

d) Calculate the MAP decision threshold η_{MAP} as a function of p_1.

e) Calculate the value of p_1 for which

e1) $\eta_{\mathrm{MAP}} = \eta_{\mathrm{ML}}$;

e2) $\eta_{\mathrm{MAP}} = 0$;

e3) $\eta_{\mathrm{MAP}} = -2/3$.

Problem 4 (20 credits)

A sequence of symbols $a[k] \in \{-1, 0, 1\}$, with $a[k] = 0$ for $k < 0$, is transmitted over a discrete-time channel with impulse response $h[k] = \delta[k] - 0.5\,\delta[k-1]$. The additive noise is white and Gaussian. The received sequence $y[k]$ is given by $y[0] = 0.8$, $y[1] = 0.2$, and $y[2] = -1.1$.

a) Represent this channel by a shift register circuit, a state diagram, and an elementary stage of the corresponding trellis diagram.

b) Use the Viterbi algorithm for ML sequence detection. Which sequence $\hat{a}[k]$ $(k = 0, 1, 2)$ is obtained with this receiver?

c) An alternative receiver uses a zero-forcing equalizer followed by a slicer. Which sequence $\hat{a}[k]$ $(k = 0, 1, 2)$ is obtained with this receiver? You may assume that $y[k] = 0$ for $k < 0$.