Digital Communications 1
Written exam on March 10, 2016

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Please note:

- You may use the official lecture notes, a pocket calculator, and a collection of mathematical formulas.
- Personal notes, materials from exercise classes, and pre-calculated problems may not be used.
- Legible writing and a clear layout of your derivations and solutions are absolutely necessary!
- Provide detailed derivations. When using results from the lecture notes, they must be explicitly referenced.

Problem 1 (20 credits)

Consider replacing the zero-forcing equalizer

$$D_{\mathrm{ZF}}(e^{j\theta}) = \frac{1}{1 - 4\cos\theta}$$

by a decision feedback equalizer. The transmit symbols of the system are taken from the alphabet $\{1, -1\}$ with equal probabilities. The symbol sequence and the equivalent discrete-time noise $Z[k]$ are uncorrelated and both white. The noise is zero-mean, and the noise variance is $\sigma_Z^2 = 1$.

a) Calculate the pulse $p[k]$ of the equivalent discrete-time baseband channel consisting of transmit pulse, physical channel, and matched receive filter.

b) Assume a general feedforward filter $d[k], k \in [-L, L]$. Calculate the overall pulse $p^{(d)}[k]$ at the output of the feedforward filter. How long is this pulse?

c) Consider now the case $L = 1$. What is the minimum length K of the feedback filter such that as much ISI as possible is canceled (under the assumption that all previous symbol decisions were correct)?

d) Calculate the coefficients of the MSE-optimum feedforward filter $\mathbf{d}_{\mathrm{MSE}}$.

Hint: The inverse of a matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

can be calculated as

$$\mathbf{A}^{-1} = \frac{1}{\det\mathbf{A}} \begin{pmatrix} a_{22}a_{33} - a_{23}a_{32} & a_{13}a_{32} - a_{12}a_{33} & a_{12}a_{23} - a_{13}a_{22} \\ a_{23}a_{31} - a_{21}a_{33} & a_{11}a_{33} - a_{13}a_{31} & a_{13}a_{21} - a_{11}a_{23} \\ a_{21}a_{32} - a_{22}a_{31} & a_{12}a_{31} - a_{11}a_{32} & a_{11}a_{22} - a_{12}a_{21} \end{pmatrix},$$

with

$$\det\mathbf{A} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

Problem 2 (20 credits)

Consider transmission of the signals

$$s^{(1)}(t) = A \sin\left(2\pi \frac{t}{T}\right) \operatorname{rect}\left(t, \frac{T}{2}\right)$$
$$s^{(2)}(t) = A \cos\left(2\pi \frac{t}{T}\right) \operatorname{rect}\left(t, \frac{T}{2}\right),$$

with $T = 2\text{ms}$, over an AWGN channel with noise power spectral density $N_0/2 = 2 \cdot 10^{-5}$ W/Hz. The transmit probabilities are given by $p_I(1) = 2/5$.

a) Find an orthonormal basis for the signal set and specify the corresponding basis coefficient vectors $\mathbf{s}^{(1)}$ and $\mathbf{s}^{(2)}$.

b) State the MAP decision rule using inner products between the basis coefficient vectors of the received signal and those of the transmit signals. Sketch a MAP sequence detector implementation that uses these inner products.

c) Since the detection problem corresponds to the binary case, the MAP sequence detector can be implemented using only one basis function. Find a basis function that is suitable for this implementation. State the MAP decision rule using only one basis coefficient of the received signal and of the transmit signals. Sketch the corresponding MAP sequence detector implementation.

d) For $A = 1.01\sqrt{\text{W}}$, calculate the conditional sequence error probabilities $\mathrm{P}_{\mathrm{MAP}}\{\mathcal{E}|I = 1\}$ and $\mathrm{P}_{\mathrm{MAP}}\{\mathcal{E}|I = 2\}$ as well as the unconditional sequence error probability $\mathrm{P}_{\mathrm{MAP}}\{\mathcal{E}\}$.

e) Find the value of A that reduces the conditional sequence error probability $\mathrm{P}_{\mathrm{MAP}}\{\mathcal{E}|I = 1\}$ to 10% of the value obtained in d).

Problem 3 (20 credits)

A random symbol $S \in \{s^{(i)}\}_{i=1}^{4}$ with

$$s^{(1)} = -4, \quad s^{(2)} = -1, \quad s^{(3)} = 1, \quad s^{(4)} = 4$$

is distributed according to

$$p_I(1) = \frac{1}{3}, \quad p_I(2) = \frac{1}{6}, \quad p_I(3) = \frac{1}{6}, \quad p_I(4) = \frac{1}{3}\,.$$

The symbol S is corrupted by additive noise N, yielding the observation

$$Y = S + N\,.$$

Here, N is statistically independent of S and distributed according to the Cauchy distribution, i.e.,

$$f_N(n) = \frac{1/\pi}{n^2+1}\,.$$

a) Calculate the probabilities $P\{N > n_1\}$ and $P\{N < n_2\}$. *Hint*: $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$.

b) Calculate the likelihood function $f_{Y|I}(y|i)$ and the posterior distribution $p_{I|Y}(i|y)$.

c) Calculate and sketch the MAP decision rule $\hat{\imath}_{\mathrm{MAP}}(y)$.

d) Calculate the conditional error probability of the MAP detector for each $i \in \{1, \ldots, 4\}$.

e) Calculate the unconditional error probability of the MAP detector.

Problem 4 (20 credits)

Consider a passband PAM system with an ML sequence detector. In the equivalent discrete-time baseband domain, the channel is described by the folded spectrum

$$S_h(z) = \frac{3z^2 + 10z + 3}{3z}.$$

Furthermore, the channel adds white Gaussian noise.

a) Sketch the function $S_h(e^{j\theta})$ for $\theta \in [-\pi, \pi]$.

b) Find the poles and zeros of $S_h(z)$.

c) Find a minimum phase factorization of $S_h(z)$.

d) Calculate the transfer function and impulse response of the equivalent discrete-time system including the noise whitening filter.

e) Consider the transmission of statistically independent symbols $a[k] \in \{1, j, -1, -j\}$ with equal probabilities. Assume that the Viterbi algorithm is to be implemented at the output of the noise whitening filter. Sketch the state transition diagram.