Digital Communications 1
Written exam on December 6, 2019

Institute of Telecommunications

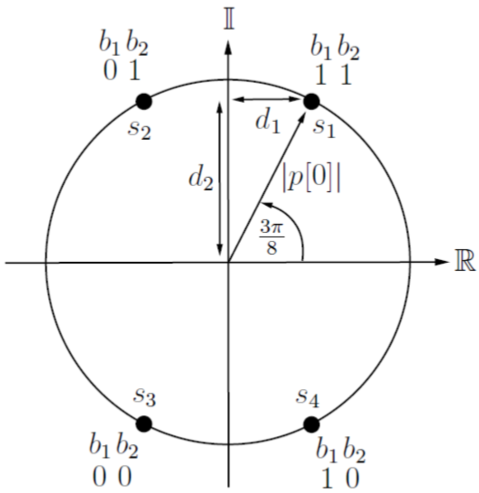
TU Wien

Please note:

- You may use the official lecture notes, a pocket calculator, and a collection of mathematical formulas.
- Personal notes, materials from exercise classes, and pre-calculated problems may not be used.
- Legible writing and a clear layout of your derivations and solutions are absolutely necessary!
- Provide detailed derivations. When using results from the lecture notes, they must be explicitly referenced.

Problem 1 (20 credits)

Asymmetric constellations provide a simple method for unequal error protection, where important bits can be protected more than bits of lesser importance. Consider an asymmetric QPSK constellation as shown in the figure below, where the mapping of two bits b_1, b_2 into each signal point is also shown. The information bits are equally likely. Communication is performed over an AWGN channel with two-sided power spectral density $\frac{N_0}{2}$. The transmission is ISI-free with gain factor $|p[0]|$.



a) Draw the decision regions of the receiver that minimizes the symbol error probability.

b) Determine the symbol error probability of the receiver in a) as a function of $\frac{|p[0]|}{\sqrt{N_0 E_f}}$, where E_f denotes the energy of the receive filter.

c) Determine the bit error probabilities for bits b_1 and b_2 separately (denoted $\mathrm{P}\{\mathcal{E}_{b_1}\}$ and $\mathrm{P}\{\mathcal{E}_{b_2}\}$, respectively). Which bit is more protected and why?

d) Assume that $N_0 = 10^{-5}$ and $E_f = 1$. How large does $|p[0]|$ need to be set to achieve $\mathrm{P}\{\mathcal{E}_{b_1}\} \leq 10^{-3}$? How large is $\mathrm{P}\{\mathcal{E}_{b_2}\}$ at that value of $|p[0]|$?

Problem 2 (20 credits)

Consider a passband PAM system with received pulse

$$h(t) = e^{-ct}\mathrm{u}(t), \qquad c > 0,$$

where $\mathrm{u}(t)$ denotes the unit step function. The additive noise is white with power spectral density $S_N(j\omega) = N_0/2$. The receiver consists of a receive filter $f(t)$, a symbol-rate sampler, and a zero-forcing (ZF) linear equalizer.

a) Assume that $f(t) = \delta(t)$, i.e., the overall complex baseband pulse is $p(t) = h(t)$.

- **a1)** Calculate the equivalent discrete-time pulse $p[k]$ at the input of the equalizer. (*Hint*: Use the discrete-time unit step function $\mathrm{u}[k]$.) Determine the variance σ_Z^2 of the equivalent discrete-time noise $Z[k]$.
- **a2)** Calculate the transfer function $D_{\mathrm{ZF}}(e^{j\theta})$ and the impulse response $d_{\mathrm{ZF}}[k]$ of the ZF linear equalizer.
- **a3)** Assume that $S_Z(e^{j\theta}) \equiv 1$. Determine the power spectral density $S_U(e^{j\theta})$ of the noise $U[k]$ at the output of the ZF linear equalizer.

b) Consider the receive filter $f(t) = h^*(-t)$.

- **b1)** Determine the transfer function $F(j\omega)$.
- **b2)** Calculate the equivalent discrete-time pulse $p[k]$ at the input of the equalizer.
- **b3)** Calculate the transfer function $D_{\mathrm{ZF}}(e^{j\theta})$ and the impulse response $d_{\mathrm{ZF}}[k]$ of the ZF linear equalizer.

Problem 3 (20 credits)

A BPSK symbol $S \in \{-1, 1\}$ with prior probability $p_S(1) = p_1$ is corrupted by additive noise $N \in \mathbb{R}$ that is statistically independent of S and distributed as follows:

$$f_N(n) = \begin{cases} \alpha(2+n) & \text{if } -2 \le n \le -1, \\ \alpha & \text{if } -1 \le n \le 0, \\ \alpha(1-n/2) & \text{if } \quad 0 \le n \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the received random variable is $Y = S + N$.

a) Sketch $f_N(n)$ and calculate α.

b) Sketch $f_{Y|S}(y|s)$ for $s \in \{-1, 1\}$. Calculate the ML decision threshold η_{ML}.

c) Calculate the conditional error probabilities $\mathrm{P}_{\mathrm{ML}}\{\mathcal{E}|S = s\}$ for $s \in \{-1, 1\}$ and the unconditional error probability $\mathrm{P}_{\mathrm{ML}}\{\mathcal{E}\}$ of the ML detector.

d) Calculate the MAP decision threshold η_{MAP} as a function of p_1.

e) Calculate the value of p_1 for which

- **e1)** $\eta_{\mathrm{MAP}} = \eta_{\mathrm{ML}}$;
- **e2)** $\eta_{\mathrm{MAP}} = 0$;
- **e3)** $\eta_{\mathrm{MAP}} = -2/3$.

Problem 4 (20 credits)

Over a discrete-time channel with impulse response $h[k] = \delta[k] - 0.5\,\delta[k-1]$, a sequence of symbols $a[k] \in \{-1, 0, 1\}$ is transmitted, with $a[k] = 0$ for $k < 0$. The additive noise is white and Gaussian. The received sequence $y[k]$ is given by $y[0] = 0.8$, $y[1] = 0.2$, and $y[2] = -1.1$.

a) Visualize this channel by a shift register circuit, a state diagram, and one stage of the corresponding trellis diagram.

b) Use the Viterbi algorithm for ML sequence detection. Which sequence $\hat{a}[k]$ $(k = 0, 1, 2)$ is obtained with this receiver?

c) An alternative receiver uses a zero-forcing equalizer followed by a slicer. Which sequence $\hat{a}[k]$ $(k = 0, 1, 2)$ is obtained with this receiver? You may assume that $y[k] = 0$ for $k < 0$.