# Modulations- und Detektionsverfahren

## Written exam on January 23, 2012

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**Please note:**

- You may use the official lecture notes, a pocket calculator, and a collection of mathematical formulas.
- Personal notes, materials from exercise classes, and pre-calculated problems may not be used.
- Legible writing and a clear layout of your derivations and solutions are absolutely necessary!
- Provide detailed derivations. When using results from the lecture notes, they must be explicitly referenced.

**Problem 1 (20 credits)**

A random variable $S \in \{s^{(i)}\}_{i=1}^{4}$ with

$$s^{(1)} = -3, \quad s^{(2)} = -1, \quad s^{(3)} = 1, \quad s^{(4)} = 3$$

is distributed according to

$$p_I(1) = \frac{1}{3}, \quad p_I(2) = \frac{1}{6}, \quad p_I(3) = \frac{1}{3}, \quad p_I(4) = \frac{1}{6}\,.$$

$S$ is corrupted by additive noise $N$, yielding the observation

$$Y = S + N\,.$$

Here, $N$ is statistically independent of $S$ and distributed according to the Cauchy distribution:

$$f_N(n) = \frac{\alpha/\pi}{n^2 + \alpha^2} \quad \text{with } \alpha > 0\,.$$

**a)** Calculate the probabilities $P\{N > n_1\}$ and $P\{N < n_2\}$. *Hint*: $\frac{d}{dx}\text{arctan}x = \frac{1}{1+x^2}$.

**b)** Calculate the value of $\alpha$ such that $P\{|N| > 1/2\} = 0.2$. Use this value of $\alpha$ in the following.

**c)** Calculate the likelihood function $f_{Y|I}(y|i)$ and the posterior distribution $p_{I|Y}(i|y)$.

**d)** Calculate and sketch the MAP decision rule $\hat{\imath}_{\text{MAP}}(y)$.

**e)** Calculate the conditional error probability of the MAP detector for each $i \in \{1, \ldots, 4\}$.

**f)** Calculate the unconditional error probability of the MAP detector.

**Problem 2 (20 credits)**

Consider a passband PAM system with an ML sequence detector. In the equivalent discrete-time baseband domain, the channel is described by the folded spectrum

$$S_h(z) = \frac{4z^2 + 17z + 4}{4z}.$$

Furthermore, the channel adds white Gaussian noise.

**a)** Sketch the function $S_h(e^{j\theta})$ for $\theta \in [-\pi, \pi]$.

**b)** Find the poles and zeros of $S_h(z)$.

**c)** Find a minimum phase factorization of $S_h(z)$.

**d)** Calculate the transfer function and impulse response of the equivalent discrete-time system including the noise whitening filter.

**e)** Consider the transmission of statistically independent symbols $a[k] \in \{1, j, -1, -j\}$ with equal probabilities. Assume that the Viterbi algorithm is to be implemented at the output of the noise whitening filter. Sketch the state transition diagram.

**Problem 3 (20 credits)**

Binary symbols $a[k] \in \{d/2, -d/2\}$ are transmitted by means of passband PAM over a dispersive channel with additive noise. The transmit symbols are equally likely. The equivalent baseband overall pulse is

$$p(t) = \alpha \cos^2\left(\frac{\pi}{2\tau}\, t\right) \operatorname{rect}\left(t; \frac{3\tau}{2}\right), \qquad \text{with } \tau = 0.5\text{ms}, \ \alpha > 0.$$

Consider a simple receiver whose symbol-rate sampler is followed directly by a slicer.

**a)** Sketch the overall pulse $p(t)$. Determine the maximum bit rate for ISI-free transmission and the corresponding symbol duration $T_s$. This bit rate is to be used in what follows.

**b)** Sketch the transmission system in the equivalent discrete-time baseband domain.

**c)** How does the output of the symbol-rate sampler, $q[k]$, depend on the transmit symbols $a[k]$?

**d)** At the slicer input, the filtered discrete-time noise is zero-mean, white, and complex Gaussian with variance $N_0$. The slicer uses the following decision rule:

$$\hat{a}[k] = \begin{cases} d/2\,, & \operatorname{Re}\{q[k]\} > 0 \\ -d/2\,, & \operatorname{Re}\{q[k]\} < 0. \end{cases}$$

What is the symbol error probability of the receiver?

**Problem 4 (20 credits)**

For two complex signals $x(t)$ and $y(t)$, consider the quantities $\|x\|$, $\|y\|$, and $\langle x, y\rangle$.

**a)** Show how these quantities are changed by the following transformations of *both* signals $x(t)$ and $y(t)$:

**a1)** a frequency shift by $\omega_0$,

**a2)** time scaling according to $x(t) \to \sqrt{|a|}\, x(at)$ and $y(t) \to \sqrt{|a|}\, y(at)$, with $a \in \mathbb{R}\backslash\{0\}$. *Hint*: use $a = |a| \cdot \operatorname{sign}(a)$.

**b)** Consider the signals

$$x(t) = A\sin(\omega_0 t)\ \operatorname{rect}(t - T; T), \qquad y(t) = A\sin(\omega_0 t + \theta)\ \operatorname{rect}(t - T; T),$$

with $T = \pi/\omega_0$ and $\omega_0 > 0$.

**b1)** Calculate $\|x\|$, $\|y\|$, and $\langle x, y\rangle$.

**b2)** Sketch $\langle x, y\rangle$ as a function of $\theta$.