Vienna University of Technology SS 2017
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Exercise Sheet 2 “Nonlinear Partial Differential Equations”
(nonlinear elliptic PDE, comparison principle, Sobolev-functions)

Exercise 1. Consider the boundary-value-problem (BVP)

$$L(u) = f \;\text{ in }\; I\,, \quad u(\alpha) = c\,, \quad u(\beta) = d\,, \tag{1}$$

for some function $f \in C^1(I)$, an interval $I = (\alpha, \beta) \subset \mathbb{R}$, constants c, $d \in \mathbb{R}$, and a quasi-linear elliptic (w.r.t u) differential operator $L(u) = a(x, u_x)u_{xx} + b(x, u, u_x)$. If the functions a, b are sufficiently smooth and the function b is monotone decreasing with respect to u then classical solutions of (1) are unique, see also Corollary 2.1 in the lecture notes. Find an example of a BVP (1) with some function b, which is not monotone decreasing in u, such that BVP (1) does not have a unique classical solution.

Exercise 2. Consider a positive constant $R > 0$ and a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary. Show that a *positive* classical solution of the BVP

$$-\Delta u = R^2 - u^2 \;\text{ in }\; \Omega\,, \qquad u = R \;\text{ on }\; \partial\Omega\,,$$

is unique.

Exercise 3. Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary and $u \in H^1(\Omega)$. Show that $u^+ \in H^1(\Omega)$ and

$$\nabla(u^+) \;=\; \begin{cases} \nabla u & \text{f.ü. auf } \{u > 0\}\,, \\ 0 & \text{f.ü. auf } \{u \le 0\}\,. \end{cases}$$

Hint: Show that $u^+ = \lim_{\epsilon\to 0} F_\epsilon(u)$, for

$$F_\epsilon(z) \;=\; \begin{cases} (z^2 + \epsilon^2)^{1/2} - \epsilon & \text{for } \; z \ge 0\,, \\ 0 & \text{for } \; z < 0\,. \end{cases}$$

Lemma 1. *Suppose $\Omega \in \mathbb{R}^n$ is a bounded domain with smooth boundary, $F \in C^1(\mathbb{R})$ with $F' \in L^\infty(\mathbb{R})$ and $u \in W^{1,p}(\mathbb{R})$ for $1 \le p \le \infty$. Then $F \circ u \in W^{1,p}(\Omega)$ and $\nabla(F \circ u) = (F' \circ u)\nabla u$.*

Exercise 4. Let X, Y, Z be Banach spaces with continuous embeddings $X \hookrightarrow Y \hookrightarrow Z$. Assume there exist $C > 0$ and $0 < \theta < 1$, s.t. for all $u \in X$:

$$\|u\|_Y \le C\|u\|_X^{1-\theta}\|u\|_Z^{\theta}.$$

Show that

(i) if $(u_k) \subset X$ is bounded with $\lim_{k\to\infty} u_k = u$ in Z then $\lim_{k\to\infty} u_k = u$ in Y.

(ii) If $X \hookrightarrow Z$ is compact, then also $X \hookrightarrow Y$ is compact.

Solutions will be discussed on Friday 16th of March 2017.