Vienna University of Technology SS 2017
Institute for Analysis and Scientific Computing
Dr. G. Di Gesù, Prof. A. Arnold

Exercise Sheet 3 "Nonlinear Partial Differential Equations"
(nonlinear elliptic PDEs, fixed point methods, weak convergence)

Exercise 1. Suppose $\Omega \in \mathbb{R}^n$ is a bounded domain with smooth boundary, $c \geq 0$ and A is a positive definite matrix with $A \geq \alpha \,\mathrm{Id}$ for some $\alpha > 0$. Consider

$$\begin{cases} L(u) := -\operatorname{div}(A\nabla u) + cu = f(x,u) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{1}$$

Moreover, suppose the function f is monotone decreasing in u, Carathéodory and satisfies

$$|f(x,u)| \leq C|u|^{p-1} + h(x) \quad \text{for almost every } x \in \Omega\,, \quad \forall u \in \mathbb{R}\,,$$

where $C \geq 0$, $p \in N^*$ and $h \in L^q(\Omega)$ with $\frac{1}{p} + \frac{1}{q} = 1$.

(i) Under which condition on the parameter C can you prove the existence of a weak solution of (1) by following the proof of Theorem 2.6 in the lecture notes?

(ii) Study the existence of a weak solution of (1) by using Schaefer's (Leray-Schauder) fixed point theorem for $X = L^p(\Omega)$.

Exercise 2. Suppose $\Omega \in \mathbb{R}^n$ is a bounded domain with smooth boundary, $c \geq 0$ and A is a positive definite matrix with $A \geq \alpha \,\mathrm{Id}$ for some $\alpha > 0$. Consider

$$\begin{cases} L(u) := -\operatorname{div}(A\nabla u) + cu = f(u) + h(x) & \text{in } \Omega\,, \\ u = g & \text{on } \partial\Omega\,, \end{cases} \tag{2}$$

with $g \in H^1(\Omega)$, $h \in L^2(\Omega)$ and $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz-continuous function. Prove the existence of a unique solution $u \in H^1(\Omega)$ of (2) with $u - g \in H_0^1(\Omega)$ in case of a sufficiently small Lipschitz-constant $Lip(f)$.

Exercise 3. Suppose $\Omega \in \mathbb{R}^n$ is a bounded domain with smooth boundary and $c \in L^\infty(\Omega)$. Consider the Dirichlet–problem

$$\begin{cases} \Delta u = e^u - c(x) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{3}$$

(i) Find a lower solution $\underline{u}(x)$, and a constant upper solution $\overline{u}$ of (3).
Hint: To construct a lower solution use the Green's function for the Laplace–operator on Ω.

(ii) Prove the existence of a solution $u \in H_0^1(\Omega) \cap H^2(\Omega)$ of (3).
Hint: Replace the nonlinear function e^u with

$$h(z) = \begin{cases} e^{\overline{u}} & \text{if } z \geq \overline{u}\,, \\ e^z & \text{if } z < \overline{u}\,, \end{cases}$$

and use Schaefer's fixed point theorem. Use a comparison principle to prove that some solutions of the modified problem are also solutions of the original problem.

(iii) Prove the uniqueness of weak solutions.

Exercise 4. Suppose $a : \mathbb{R} \to \mathbb{R}$ is a continuous function, such that $a(f_n) \rightharpoonup a(f)$ converges weakly in $L^2(0,1)$ for all weakly convergent sequences $f_n \rightharpoonup f$ in $L^2(0,1)$. Prove that a is an affine function, i.e. the function a has for suitable constants α and β the representation $a : z \mapsto a(z) = \alpha z + \beta$ for all $z \in \mathbb{R}$.

Hint: Suppose $a(su + (1-s)v) \neq sa(u) + (1-s)a(v)$ for s, u, v, $(su + (1-s)v) \in (0,1)$. Find a contradiction using the sequence

$$f_n(z) = \begin{cases} u & \text{if } z \in [j/n,\, (j+s)/n] \text{ mit } j = 0, \ldots, n-1\,, \\ v & \text{otherwise,} \end{cases}$$

and simple testfunctions.

Solutions will be discussed on Monday 27th of March 2017.