Vienna University of Technology SS 2017
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Exercise Sheet 6 "Nonlinear Partial Differential Equations"
(parabolic PDEs, reaction-diffusion equations)

Exercise 1. Let $\Omega \subset \mathbb{R}^n$ be bounded, let $T > 0$, and let $X := C([0,T]; L^2(\Omega))$. Consider the map $A : X \to X$, $v \mapsto u$, which assigns to $v \in X$ the unique weak solution u of

$$\begin{cases} u_t - \Delta u = f(v) & \text{in } \Omega \times (0,T]\,, \\ u = 0 & \text{for } (x,t) \in \partial\Omega \times [0,T]\,, \\ u(t=0,\cdot) = u_0 & \text{in } \Omega\,, \end{cases} \tag{1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz-continuous and $u_0 \in L^2(\Omega)$. Show that A^k is contractive for sufficiently large $k = k(T)$.

Exercise 2. Let u be the weak solution of the Cauchy-Problem

$$\begin{cases} u_t - \Delta u = f(u) & \text{in } \Omega \times (0,\infty)\,, \\ u = 0 & \text{for } (x,t) \in \partial\Omega \times [0,\infty)\,, \\ u(t=0,\cdot) = u_0 & \text{in } \Omega\,, \end{cases} \tag{2}$$

where $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz-continuous and $u_0 \in L^2(\Omega)$. Give an estimate for the norm $\|u(t,\cdot)\|_{L^2(\Omega)}$, valid for every $t \geq 0$.

Exercise 3. Provide examples for functions $f : \mathbb{R} \to \mathbb{R}$ and initial values $u_0 \in L^2(\Omega)$, such that the Cauchy-Problem

$$\begin{cases} u_t - \Delta u = f(u) & \text{in } \Omega \times (0,\infty)\,, \\ u = 0 & \text{for } (x,t) \in \partial\Omega \times [0,\infty)\,, \\ u(t=0,\cdot) = u_0 & \text{in } \Omega\,, \end{cases} \tag{3}$$

admits local but not global solutions and prove your claims.

Exercise 4. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$ and let $T > 0$. Consider the differential operator

$$Lu = -\operatorname{div}(A(x,t)\nabla u) + b(x,t) \cdot \nabla u + c(x,t)u\,, \tag{4}$$

with strictly positive definite matrix $A = (a^{ij})_{i,j=1,\ldots,n}$, a vector $b = (b^i)_{i=1,\ldots,n}$, and bounded continuous coefficients a^{ij}, b^i, $c \in C_b(\Omega \times [0,T])$ for $i,j = 1,\ldots,n$.

Consider the initial-value-problem (IVP)

$$\begin{cases} u_t + Lu = f & \text{in } \Omega \times (0,T]\,, \\ u(x,0) = u_0(x) & \text{for } x \in \Omega\,, \end{cases} \tag{5}$$

with the differential operator L in (4), $u_0 \in L^2(\Omega)$, $f \in L^2(\Omega \times (0,T])$. Prove the existence of a unique weak solution u of the IVP (5) with Dirichlet-boundary condition

$$u(x,t) = 0 \text{ for } (x,t) \in \partial\Omega \times [0,T]\,.$$

Solutions will be discussed on Monday 8th of May 2017.