Vienna University of Technology SS 2017
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Exercise Sheet 8 "Nonlinear Partial Differential Equations"
(parabolic PDEs)

Exercise 1. Let V be a reflexive, separable Banach-space and $X := L^p(0,T;V)$ for some $p \in (1,\infty)$. Consider an operator $A : V \to V'$ which has a space-time-interpretation $A : X \to X'$. Show that

(i) A is V-monotone if and only if A is X-monotone.

(ii) A bounded operator A satisfying $\|Av\|_{V'} \leq C\|v\|_V^{p-1}$ for all $v \in V$ is V-hemi-continuous (resp. V-demi-continuous) if and only if A is X-hemi-continuous (resp. X-demi-continuous).

Exercise 2. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and $T > 0$. Consider the operator $A : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ defined as

$$A(u) := -\operatorname{div}(a(u)\nabla u),$$

where $a \in C(\mathbb{R})$ and $0 < \delta_1 \leq a(u) \leq \delta_2$. Show that

(i) $A(u(\cdot)) : [0,T] \longrightarrow H^{-1}(\Omega)$ is measurable for all $u \in X = L^2(0,T;H_0^1(\Omega))$;

(ii) $A : X \longrightarrow X'$;

(iii) $A : X \longrightarrow X'$ is bounded;

(iv) $A : X \longrightarrow X'$ is coercive.

Exercise 3. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, $2 \leq p < \infty$ and $T > 0$. Consider the initial/boundary-value problem

$$\begin{cases} u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 & \text{in } \Omega \times (0,T)\,, \\ u = 0 & \text{on } \partial\Omega \times [0,T]\,, \\ u(\cdot, t=0) = u_0 & \text{in } \Omega\,, \end{cases} \tag{1}$$

with $u_0 \in L^2(\Omega)$. Show that a unique weak solution $u \in C([0,T];L^2(\Omega))$ exists.

Exercise 4. Let $\Omega = \mathbb{R}^3$ and $T > 0$. Consider the Navier-Stokes-equation

$$u_t + (u \cdot \nabla)u + \nabla p = \Delta u \text{ in } \Omega \times (0,T]\,. \tag{2}$$

Show that the Navier-Stokes equation supports the following symmetry groups:

(i) Galilean invariance: If (u,p) is a solution of (2) and $c \in \mathbb{R}^3$ is a constant vector, then

$$u_c(x,t) := u(x-ct,t) + c\,, \quad p_c(x,t) := p(x-ct,t)\,,$$

is a solution of (2).

(ii) Rotation symmetry: If (u,p) is a solution of (2) and $Q \in \mathbb{R}^{3\times 3}$ is a rotation matrix, i.e. $Q^T = Q^{-1}$, then

$$u_Q(x,t) := Q^T u(Qx,t)\,, \quad p_Q(x,t) := p(Qx,t)\,,$$

is a solution of (2).

(iii) Scale invariance: If (u,p) is a solution of (2) and $\tau \in \mathbb{R}_+$, then

$$u_\tau(x,t) := \tfrac{1}{\sqrt{\tau}} u\big(\tfrac{x}{\sqrt{\tau}}, \tfrac{t}{\tau}\big)\,, \quad p_\tau(x,t) := \tfrac{1}{\tau} p\big(\tfrac{x}{\sqrt{\tau}}, \tfrac{t}{\tau}\big)\,,$$

is a solution of (2).

Solutions will be discussed on Monday 22th of May 2017.