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Exercise Sheet 9 "Nonlinear Partial Differential Equations"
(incompressible Navier-Stokes equation, porous medium equation)

Exercise 1. Let $n \in \{2,3\}$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Consider the initial/boundary-value problem for the incompressible NAVIER-STOKES-equation

$$\begin{cases} u_t - \Delta u + (u \cdot \nabla)u + \nabla p = 0 & \text{in } \Omega \times (0,\infty)\,, \\ \operatorname{div} u = 0 & \text{in } \Omega \times (0,\infty)\,, \\ u = 0 & \text{on } \partial\Omega \times [0,\infty)\,, \\ u(\cdot,0) = u_0 & \text{in } \Omega\,, \end{cases} \tag{1}$$

with $u_0 \in L^2(\Omega)$. Show that a classical solution (u,p) satisfies the estimate

$$\|u(\cdot,t)\|_{L^2(\Omega)} \le e^{-\lambda t}\|u_0\|_{L^2(\Omega)}$$

for some constant $\lambda > 0$ and for all $t > 0$.

Exercise 2. Consider for each $m > 1$ the function

$$U_m : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}\,, \quad (x,t) \mapsto t^{-\lambda}\left(C - K\frac{|x|^2}{t^{\frac{2\lambda}{n}}}\right)_+^{\frac{1}{m-1}}$$

with $C > 0$, $\lambda = \frac{n}{n(m-1)+2}$ and $K = \frac{\lambda(m-1)}{2mn}$.

Remark: The function U is known as Barenblatt-solution of the porous medium-equation

$$u_t - \Delta_x(u^m) = 0 \quad \text{in} \quad \mathbb{R}^n \times (0,\infty)\,. \tag{2}$$

(i) Show that $(0,\infty) \ni t \mapsto \int_{\mathbb{R}^n} U_m(x,t)dx$ is constant and that $U(\cdot,t) \to M\delta$ for $t \to 0$, where δ is the Delta-distribution and $M := \int_{\mathbb{R}^n} U_m(x,t)dx$.

(ii) Assume that U_m is normalized, so that its mass M equals one: what do you expect to be the limit behaviour of U_m for $m \to 1$?

Prove that your guess is true. You can use without proof that $M = 1$ if $C = D^\gamma$, with $D = \frac{1}{2}K^{-\frac{n}{2}}\, n\omega_n\, B(\frac{n}{2}, \frac{m}{m-1})$ and $\gamma = \frac{n}{2(m-1)\lambda}$ (here ω_n denotes the volume of the unit ball in dimension n and B the Euler beta function).

Exercise 3. Let $m > 1$ and let U_m be the function defined in the previous exercise.

(i) Discuss the regularity of U_m.

(ii) Show that $U_m(x, t+\tau)$, for a fixed $\tau > 0$, is a weak solution of the *porous medium-*equation

$$u_t - \Delta_x(u^m) = 0 \quad \text{in} \quad \mathbb{R}^n \times (0,\infty)\,. \tag{3}$$

Exercise 4. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary. Let $0 \le u_0 \in C_0^\infty(\Omega)$ and let u_n with $n \in \mathbb{N}$ be solutions of

$$\begin{cases} (u_n)_t = \Delta u_n^m & \text{in } \Omega \times (0,\infty)\,, \\ u_n(x,t) = \frac{1}{n} & \text{on } \partial\Omega \times [0,\infty)\,, \\ u_n(x,0) = u_0 + \frac{1}{n} & \text{in } \Omega\,. \end{cases} \tag{4}$$

Show that $0 \le u_{n+1}(x,t) \le u_n(x,t)$ in $\Omega \times (0,\infty)$ for all $n \in \mathbb{N}$.

Solutions will be discussed on Wednesday 24th of May 2017.