Vienna University of Technology SS 2019
Institute for Analysis and Scientific Computing
Dr. B. Nectoux, Prof. A. Arnold

Exercise Sheet 3 "Nonlinear Partial Differential Equations"
(nonlinear elliptic PDEs, fixed point methods, weak convergence)

Exercise 1. Suppose $\Omega \in \mathbb{R}^n$ is a bounded domain with smooth boundary, c is a non negative function in Ω and A is a positive definite matrix with $A \geq \alpha\,\mathrm{Id}$ for some $\alpha > 0$. Let us consider the following non linear PDE,

$$\begin{cases} L(u) := -\operatorname{div}(A\nabla u) + cu = f(x,u) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{1}$$

Moreover, suppose the function f is Carathéodory and satisfies

$$|f(x,u)| \leq C_f |u|^{p-1} + h(x) \qquad \text{for almost every } x \in \Omega\,, \quad \forall u \in \mathbb{R}\,,$$

where $C_f \geq 0$, $p \in \mathbb{N}^*$ and $h \in L^q(\Omega)$ with $\frac{1}{p} + \frac{1}{q} = 1$.

(i) Recall the definition of a weak solution of (1). Check that all the terms appearing in the weak formulation are well defined.

(ii) Under which condition on the parameter C_f can you prove the existence of a weak solution of (1) by following the proof of Theorem 2.12 in the lecture notes?

(iii) Let us now assume in addition that f is monotone decreasing in u. Show the existence of a weak solution of (1) by using Schaefer's (Leray-Schauder) fixed point theorem for $X = L^p(\Omega)$.

Do we still need a condition on C_f?

(iv) Let us still assume that f is monotone decreasing in u. Study the uniqueness of a weak solution of (1).

Exercise 2. Suppose $\Omega \in \mathbb{R}^n$ is a bounded domain with smooth boundary, $c \geq 0$ is a constant and A is a positive definite matrix with $A \geq \alpha\,\mathrm{Id}$ for some $\alpha > 0$. Consider

$$\begin{cases} L(u) := -\operatorname{div}(A\nabla u) + cu = f(u) + h(x) & \text{in } \Omega\,, \\ u = g & \text{on } \partial\Omega\,, \end{cases} \tag{2}$$

with $g \in H^1(\Omega)$, $h \in L^2(\Omega)$ and $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz-continuous function. Prove the existence of a unique solution $u \in H^1(\Omega)$ of (2) with $u - g \in H_0^1(\Omega)$ in case of a sufficiently small Lipschitz-constant $Lip(f)$.

Exercise 3. Suppose $\Omega \in \mathbb{R}^n$ is a bounded domain with smooth boundary and $c \in L^\infty(\Omega)$. Consider the Dirichlet–problem

$$\begin{cases} \Delta u = e^u - c(x) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{3}$$

(i) Find constant upper solution $\overline{u}$ of (3). With this upper solution, construct a lower solution $\underline{u}(x)$. Then, recall the comparison principle for weak solutions.
Hint: Construct $\underline{u}(x)$ as a solution of:

$$\Delta \underline{u} = f \text{ in } \Omega \quad \text{with } \underline{u} = 0 \text{ on } \partial\Omega,$$

for a well chosen f.

(ii) Prove the existence of a weak solution $w \in H^2(\Omega) \cap H_0^1(\Omega)$ of

$$\begin{cases} \Delta u = h(u) - c(x) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases}$$

where

$$h(z) := \begin{cases} e^{\overline{u}} & \text{if } z \geq \overline{u}\,, \\ e^z & \text{if } z < \overline{u}\,, \end{cases}$$

Hint: Item (iii) in the first exercise.
Show that some solutions of this modified problem are also solutions of the original problem.

(iii) Prove the uniqueness of weak solutions.

Exercise 4. Suppose $a : \mathbb{R} \to \mathbb{R}$ is a continuous function, such that $a(f_n) \rightharpoonup a(f)$ converges weakly in $L^2(0,1)$ for all weakly convergent sequences $f_n \rightharpoonup f$ in $L^2(0,1)$. Prove that a is an affine function, i.e. the function a has for suitable constants α and β the representation $a : z \mapsto a(z) = \alpha z + \beta$ for all $z \in \mathbb{R}$.

Hint: Suppose $a(su + (1-s)v) \neq sa(u) + (1-s)a(v)$ for s, u, v, $(su + (1-s)v) \in (0,1)$. Find a contradiction using the sequence

$$f_n(z) = \begin{cases} u & \text{if } z \in [j/n,\, (j+s)/n] \text{ mit } j = 0, \dots, n-1\,, \\ v & \text{otherwise,} \end{cases}$$

and simple testfunctions.

Solutions will be discussed on Monday 28th of March 2019.