Vienna University of Technology SS 2019
Institute for Analysis and Scientific Computing
Dr. B. Nectoux, Prof. A. Arnold

# Exercise Sheet 4 "Nonlinear Partial Differential Equations"

(nonlinear elliptic PDE, weak convergence, monotone operators)

---

**Theorem 1.** *Let $n \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary $\partial\Omega \in C^1$.*

*(i) There exists a bounded linear operator, the so called* trace-*operator, $T : H^1(\Omega) \to L^2(\partial\Omega)$ with the property $T(u) = u|_{\partial\Omega}$ for all $u \in H^1(\Omega) \cap C^0(\overline{\Omega})$.*

*(ii) If $u \in H^1(\Omega)$, then $u \in H_0^1(\Omega)$ if and only if $T(u) = 0$ in $L^2(\Omega)$.*

**Exercise 1.** Let $\Omega \in \mathbb{R}^n$ be a bounded domain with smooth boundary. Consider the boundary value problem (BVP) for a nonlinear POISSON-equation

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1}$$

where $f \in C^1(\mathbb{R})$ with $|f'| \leq C$. Prove the following statement:

If there exist a weak upper solution $\overline{u}$ and a weak lower solution $\underline{u}$ of BVP (1) such that

$$T(\underline{u}) \leq 0 \text{ on } \partial\Omega\,, \quad T(\overline{u}) \geq 0 \text{ on } \partial\Omega\,, \quad \underline{u} \leq \overline{u} \text{ almost everywhere in } \Omega\,,$$

then there exists a weak solution $u$ of BVP (1) such that

$$\underline{u} \leq u \leq \overline{u} \quad \text{almost everywhere in } \Omega\,.$$

Hint:

(i) Prove, for a suitable $\lambda > 0$, the existence of a sequence of weak solutions $(u_k)_{k\in\mathbb{N}}$ for the linear BVPs

$$\begin{cases} -\Delta u_{k+1} + \lambda u_{k+1} = f(u_k) + \lambda u_k & \text{in } \Omega\,, \\ u_{k+1} = 0 & \text{on } \partial\Omega\,, \end{cases}$$

which satisfy the estimates

$$\underline{u} = u_0 \leq u_1 \leq \cdots \leq u_k \leq \cdots \leq \overline{u} \quad \text{a.e. in } \Omega\,.$$

(ii) Prove the existence of the limit $u := \lim_{k\to\infty} u_k$ in $L^2(\Omega)$ and of a subsequence $(u_{k_j})_{j\in\mathbb{N}}$ such that $u_{k_j} \rightharpoonup u$ in $H_0^1(\Omega)$. Use these results in the weak formulation of BVP (1), to recover $u$ as a weak solution of BVP (1).

**Exercise 2.** Suppose $f \in C^0(\mathbb{R})$ is a (not necessarily strongly) monotone function such that $f(w) \in L^2(\Omega)$ for all $w \in L^2(\Omega)$, and $(u_k)_{k\in\mathbb{N}} \subset L^2(\Omega)$ is a sequence with $u_k \to u$ in $L^2(\Omega)$ for $k \to \infty$. Use the method of BROWDER and MINTY, to prove

$$f(u_k) \rightharpoonup v \text{ in } L^2(\Omega) \text{ for } k \to \infty \quad \Rightarrow \quad v = f(u)\,.$$

**Exercise 3.** Suppose $\Omega \Subset \mathbb{R}^n$ is a bounded domain with smooth boundary. Consider the *minimal surface equation*

$$\begin{cases} \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = f & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{2}$$

Show that there exists at most one weak solution of (2).

Hint: First show for all $p$, $q \in \mathbb{R}^n$ that

$$\left(\frac{p}{\sqrt{1+|p|^2}} - \frac{q}{\sqrt{1+|q|^2}}\right)\cdot(p-q) \geq \frac{\sqrt{1+|p|^2}+\sqrt{1+|q|^2}}{2}\left|\frac{p}{\sqrt{1+|p|^2}} - \frac{q}{\sqrt{1+|q|^2}}\right|^2.$$

Then show $u = v$ in $\Omega$ for all weak solutions $u$, $v \in H^1_0(\Omega)$ of (2).

**Theorem 2.** *Suppose $B$ is a reflexive BANACH space. If all weakly convergent subsequences of a bounded sequence $\{\, u_k \,|\, k \in \mathbb{N} \,\} \subset B$ converge weakly to the same limit $u \in B$, then the whole sequence $(u_k)_{k\in\mathbb{N}}$ converges weakly to $u$.*

**Exercise 4.** Suppose $V$ is a reflexive BANACH space and $A : V \to V'$. Show that

(i) If $A$ is hemi-continuous and monotone, then $A$ is of type M.

(ii) If $A$ is of type M and bounded, then $A$ is demi-continuous.

(iii) If $A$ is (strongly) continuous, then $A$ is demi-continuous. If $A$ is demi-continuous, then $A$ is hemi-continuous.

(iv) If $A$ is strongly continuous, then $A$ is compact.

Solutions will be discussed on Thursday 4th of April 2019.