Vienna University of Technology SS 2019
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Exercise Sheet 5 "Nonlinear Partial Differential Equations"
(nonlinear elliptic PDE, monotone operators, GALERKIN-approximation, HELMHOLTZ-decomposition)

Exercise 1. Suppose V is a reflexive BANACH space, $(v_k)_{k\in\mathbb{N}}$ is a basis of V, $f \in V'$, and $A : V \to V'$ is a bounded and strongly monotone operator of type M.
For $m \in \mathbb{N}$, let $u_m \in V_m = \operatorname{span}\{v_1, \dots, v_m\}$ be a solution of

$$\langle A(u_m), v_k\rangle_{V'} = \langle f, v_k\rangle_{V'}, \quad k = 1, \dots, m\,,$$

and assume that

$$\exists C > 0 \,:\, \|u_m\|_V \le C \quad \forall m \in \mathbb{N}\,.$$

Show that: There exists a $u \in V$ such that the sequence (u_m) converges to u in V and u is the unique solution of $A(u) = f$.

Exercise 2. Suppose $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, $f \in (L^2(\Omega))^3$ and $X := \{u \in H_0^1(\Omega)^3 \,|\, \operatorname{div} u = 0 \text{ a.e. in } \Omega\}$. Consider the stationary STOKES-equation

$$\begin{cases} \nabla p - \Delta u = f & \text{in } \Omega\,, \\ \operatorname{div} u = 0 & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{1}$$

(i) Determine a weak formulation of (1) and prove the existence of a unique weak solution $u \in X$.

(ii) How can you determine p from a weak solution $u \in X$?

Exercise 3. Let $n \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. Consider the vector-space

$$H(\operatorname{div}, \Omega) := \{\, u \in L^2(\Omega)^n \,|\, \operatorname{div} u \in L^2(\Omega) \,\}\,. \tag{2}$$

Show that

(i) The map $(\cdot,\cdot)_{H(\operatorname{div},\Omega)} : H(\operatorname{div},\Omega) \times H(\operatorname{div},\Omega) \to \mathbb{C}$, defined as

$$(u, v) \mapsto (u, v)_{H(\operatorname{div},\Omega)} := (u, v)_{L^2(\Omega)^n} + (\operatorname{div} u, \operatorname{div} v)_{L^2(\Omega)}\,, \text{ is a scalar-product.}$$

(ii) The vector-space $H(\operatorname{div}, \Omega)$ with scalar-product $(\cdot,\cdot)_{H(\operatorname{div},\Omega)}$ is a HILBERT-space.

Exercise 4. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$, and

$$H_0(\operatorname{div} 0, \Omega) := \{u \in L^2(\Omega)^3 | \operatorname{div} u = 0 \text{ a.e. in } \Omega, T_\nu(u) = 0\}.$$

Prove the HELMHOLTZ-decomposition: $(L^2(\Omega))^3 = H_0(\operatorname{div} 0, \Omega) \oplus \nabla H^1(\Omega)$.
Hint: Use

EITHER $\nabla H^1(\Omega)$ is a closed subset of $L^2(\Omega)^3$.

OR Consider $\Delta p = \operatorname{div} u \quad \text{in } \Omega, \quad T_\nu(\nabla p) = 0 \quad \text{on } \partial\Omega$, for $u \in L^2(\Omega)^3$ and study the existence of weak solutions p in a subspace of $H^1(\Omega)$.

The following result can be used without proof:

Theorem 1. *Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary $\partial\Omega \in C^2$, $T : H^1(\Omega) \to L^2(\partial\Omega)$ be the standard trace operator, and $\nu(x)$ be the (outward pointing) unit normal vector at $x \in \partial\Omega$.*

(i) There exists a continuous linear operator $T_\nu \in \mathcal{L}(H(\operatorname{div}, \Omega), T(H^1(\Omega))')$, such that

$$T_\nu u := (u \cdot \nu)|_{\partial\Omega} \text{ for all } u \in \mathcal{D}(\overline{\Omega}),$$

and the STOKES formula holds for all $u \in H(\operatorname{div}, \Omega)$ and $w \in H^1(\Omega)$:

$$(u, \nabla w) + (\operatorname{div} u, w) = \langle T_\nu u, Tw \rangle.$$

(ii) Suppose $H_0(\operatorname{div}, \Omega)$ is the closure of $\mathcal{D}(\Omega)^n$ in $H(\operatorname{div}, \Omega)$. If $u \in H(\operatorname{div}, \Omega)$, then $u \in H_0(\operatorname{div}, \Omega)$ if and only if $T_\nu(u) = 0$ in $T(H^1(\Omega))'$.

Solutions will be discussed on Thursday 11th of April 2019.