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Exercise Sheet 11 "Nonlinear Partial Differential Equations"
(Nonlinear functionals, wave equation)

Consider a functional $\mathcal{F} : U \to \mathbb{R}$ where $U \subset X$ and X is a linear space over $\mathbb{R}$. If $u_0 \in U$, $\xi \in X$ and $\epsilon_0 > 0$ are such that $\{ u_0 + \epsilon\xi \,|\, |\epsilon| < \epsilon_0 \} \subset U$, then the function

$$\phi : (-\epsilon_0, \epsilon_0) \to \mathbb{R}\,, \quad \epsilon \mapsto \phi(\epsilon) := \mathcal{F}(u_0 + \epsilon\xi)\,,$$

is welldefined.

Definition 1. If $\phi'(0)$ exists, then $\delta\mathcal{F}(u_0, \xi) := \phi'(0)$ is called the *first variation* of $\mathcal{F}$ at u_0 in direction of ξ.

Remark 2. The "variation" is a "weak" derivative concept, which does not need a topology on X.

Definition 3. Let X be a Banach-space and X' its topological dual space. Let $\mathcal{F} : U \to \mathbb{R}$ be a functional for some open set $U \subset X$ and $u_0 \in U$.

(i) The functional $\mathcal{F}$ is called Gâteaux differentiable at u_0 if there exists $l \in X'$ (depending on u_0) such that

$$\lim_{\epsilon \to 0} \left| \frac{\mathcal{F}(u_0 + \epsilon\xi) - \mathcal{F}(u_0) - \epsilon l(\xi)}{\epsilon} \right| = 0 \quad \text{for all } \xi \in X\,.$$

In this case $\mathrm{d}\mathcal{F}(u_0, \cdot) = l$ is called the Gâteaux *derivative* of $\mathcal{F}$ at u_0.

(ii) The functional $\mathcal{F}$ is called Fréchet differentiable at u_0 if there exists $l \in X'$ (depending on u_0) such that

$$\lim_{\xi \to 0} \frac{|\mathcal{F}(u_0 + \xi) - \mathcal{F}(u_0) - l(\xi)|}{\|\xi\|} = 0 \quad \text{for all } \xi \in X\,.$$

In this case $\mathrm{D}\mathcal{F}(u_0, \cdot) = l$ is called the Fréchet *derivative* of $\mathcal{F}$ at u_0.

Exercise 1. Let X be a Banach-space and X' be its topological dual space. Consider the functional $\mathcal{F} : U \to \mathbb{R}$ on some open subset $U \subset X$ and $u_0 \in U$. Show that

(i) $\mathcal{F}$ Fréchet differentiable at $u_0 \Rightarrow \mathcal{F}$ Gâteaux differentiable at $u_0 \Rightarrow$ First variation of $\mathcal{F}$ exists in all directions ξ.

(ii) $\mathcal{F}$ Fréchet differentiable at $u_0 \Rightarrow$

$$\mathrm{D}\mathcal{F}(u_0, \xi) = \mathrm{d}\mathcal{F}(u_0, \xi) = \delta\mathcal{F}(u_0, \xi) \quad \text{for all } \xi \in X\,.$$

(iii) How are Gâteaux derivative resp. Fréchet derivative called in a finite-dimensional Banach-space X?

Exercise 2. Let $X := H^2(\mathbb{R}^n_x \times \mathbb{R}_t)$. Consider the functionals

(i) $\mathcal{E}_1 : X \to \mathbb{R}$, $u \mapsto \mathcal{E}_1(u) := \int_{\mathbb{R}^n_x \times \mathbb{R}_t} -\frac{1}{2}u_t^2 + \frac{1}{2}|\nabla u|^2 + F(u)\, d(x,t)$, where $F \in C^1(\mathbb{R};\mathbb{R})$ and $F : X \to L^1(\mathbb{R}^n_x \times \mathbb{R}_t)$.

(ii) $\mathcal{E}_2 : X \to \mathbb{R}$, $u \mapsto \mathcal{E}_2(u) := \int_{\mathbb{R}^n_x \times \mathbb{R}_t} \frac{1}{2}\mathrm{Im}(u_t \bar{u}) + \frac{1}{2}|\nabla u|^2 + F(u)\, d(x,t)$, where $F \in C^1(\mathbb{C};\mathbb{R})$, $F : X \to L^1(\mathbb{R}^n_x \times \mathbb{R}_t)$, and $\bar{u}$ is the complex-conjugate of u.

Compute the *first variation* and the GÂTEAUX derivative of $\mathcal{E}_1$ and $\mathcal{E}_2$.

Exercise 3. Consider the evolution group $T_0(t) := \mathrm{e}^{\mathrm{i}t\Delta}$, $t \in \mathbb{R}$, for the free SCHRÖDINGER equation on $L^2(\mathbb{R}^n)$ with

$$T_0(t)u_0 := \begin{cases} u_0 & \text{for } t = 0\,, \\ u(x,t) = \int_{\mathbb{R}^n} u_0(\xi) \frac{1}{(4\pi \mathrm{i} t)^{n/2}} \mathrm{e}^{\frac{\mathrm{i}|x-\xi|^2}{4t}}\, \mathrm{d}\xi & \text{for } t \neq 0\,, x \in \mathbb{R}^n\,. \end{cases}$$

Show that all operators $T_0(t)$, $t \in \mathbb{R}$, are unitary.

Exercise 4. Show that $\int (xu + 3tu^2) dx$ is a conserved quantity for the KdV equation

$$u_t = 6uu_x - u_{xxx}\ .$$

Solutions will be discussed on Thursday 13th of June 2019.