Vienna University of Technology SS 2019
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Exercise Sheet 12 "Nonlinear Partial Differential Equations"

Exercise 1. Consider the nonlinear SCHRÖDINGER-equation

$$\mathrm{i}\psi_t + \psi_{xx} + \psi|\psi|^2 = 0 \quad \text{for} \quad (x,t) \in \mathbb{R}\times\mathbb{R}\,.$$

(i) Search for soliton-solutions of the form $\psi = r\mathrm{e}^{\mathrm{i}(\theta+nt)}$ with real-valued functions $r(x-ct)$ and $\theta(x-ct)$, where c and n are real constants. Derive ordinary differential equations for θ and r.

(ii) Show that, for $S = r^2$ and $F(S) = S^3 - 2(n - \frac{1}{4}c^2)S^2 + BS + \frac{1}{2}A^2$ with arbitrary real integration constants A and B,

$$\theta' = \tfrac{1}{2}(c + \tfrac{A}{S}) \quad \text{and} \quad (S')^2 = -2F(S)$$

hold.

(iii) Consider the special case $A = B = 0$. Find the roots of $F(S)$ and the corresponding periodic solutions ψ.

(iv) Show that a 'solitary-wave'-solution of the form

$$\psi(x,t) = a\mathrm{e}^{\mathrm{i}(\frac{1}{2}c(x-ct)+nt)}\big(\cosh(\tfrac{a(x-ct)}{\sqrt{2}})\big)^{-1}$$

exists.

Exercise 2. Suppose $R > 0$, $1 < p < \infty$ and $r > 2$. Consider the subset

$$Y := \big\{u \in C_t(\mathbb{R}; L^2_x(\mathbb{R}^n)) \cap L^r_t(\mathbb{R}; L^{p+1}_x(\mathbb{R}^n)) \mid \|u\|_{L^\infty_t(H^1_x)} + \|u\|_{L^r_t(W^{1,p+1}_x)} \le R\big\}$$

of the BANACH-space $C_t(L^2_x) \cap L^r_t(L^{p+1}_x)$ with norm $\|u\|_{L^\infty_t(L^2_x)} + \|u\|_{L^r_t(L^{p+1}_x)}$. Show that, Y is a closed subset of $C_t(L^2_x) \cap L^r_t(L^{p+1}_x)$, although Y is defined via a stronger norm.

Hint: use the following theorem.

Theorem 1 (BANACH-ALAOGLU-BOURBAKI)**.** *Let X be a normed space, the dual X' is hence also a normed space (with the operator norm). Then the closed unit ball of X' is compact with respect to the weak* topology.*

Exercise 3. Show that the SCHRÖDINGER-POISSON-equation

$$\begin{cases} i\psi_t + \Delta\psi + f(\psi) = 0 & \text{for } (x,t) \in \mathbb{R}^3 \times \mathbb{R}\,, \\ \psi(t=0) = \psi_0 \in H^1(\mathbb{R}^3)\,, \end{cases}$$

with $f(\psi) = \frac{1}{4\pi}(|\psi|^2 * \frac{1}{|x|})\psi$, has a unique mild solution $\psi \in C(\mathbb{R}, H^1(\mathbb{R}^3))$.

Remark: $V[\psi] := \frac{1}{4\pi}(|\psi|^2 * \frac{1}{|x|})$ solves the POISSON-equation $-\Delta V = |\psi|^2$.

Hint: Show via the generalized YOUNG-inequality (Lemma 4.5) and the SOBOLEV-imbedding $W^{1,p}(\mathbb{R}^n) \hookrightarrow C_B(\mathbb{R}^n)$ for $p > n$, that $f : H^1(\mathbb{R}^3) \to H^1(\mathbb{R}^3)$ is a local LIPSCHITZ-continuous function. Use Theorem 4.6. Finally, show that

$$\|\psi(t)\|_{L^2(\mathbb{R}^3)} \quad \text{and} \quad E(\Psi(t)) = \|\nabla\psi(t)\|^2_{L^2(\mathbb{R}^3)} + \tfrac{1}{2}\|\nabla V(t)\|^2_{L^2(\mathbb{R}^3)}$$

are conserved quantities and deduce the necessary a-priori-estimates.

Solutions will be discussed on Thursday 27th of June 2019.