Vienna University of Technology SS 2017
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Exercise Sheet 1 "Nonlinear Partial Differential Equations"
(Separation of variables, Blow-up, Method of characteristics, Nonlinear transformation)

Exercise 1. Consider the *porous medium*–equation

$$u_t - \Delta_x(u^\gamma) \;=\; 0 \quad \text{in} \quad \mathbb{R}^n \times (0,\infty) \ni (x,t)\,, \tag{1}$$

for a constant $\gamma > 1$ and a non-negative scalar function $u \geq 0$.

(i) Find a solution of equation (1) via a *separation of variables*, i.e. consider the ansatz $u(x,t) = v(t)w(x)$. Hint: Use $w(x) = |x|^\alpha$ for some $\alpha > 0$.

(ii) Find a scaling-invariant solution $u(x,t)$ in the form $u(x,t) = \frac{1}{t^\alpha} f\big(\frac{|x|}{t^\beta}\big)$ for some constants α, $\beta \in \mathbb{R}$ with $\alpha + 1 = \alpha\gamma + 2\beta$ and a function $f : \mathbb{R} \to \mathbb{R}$.

Sketch your solutions and discuss their behavior for $t \to \infty$.

Exercise 2. Suppose $T > 0$ and $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary. Consider the initial-value-problem (IVP) for a reaction-diffusion equation

$$\begin{cases} u_t = \Delta u + f(u) & \text{in } G := \Omega \times (0,T]\,, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T)\,, \\ u(\cdot,0) = u_0 & \text{in } \Omega\,, \end{cases} \tag{2}$$

and continuous functions $f : \mathbb{R} \to \mathbb{R}$ and $u_0 \in C(\overline{\Omega})$.

(i) Compute for $f(u) = u^2$ a spatial homogeneous solution $u = u(t)$ of (2) and discuss its asymptotic behavior for $t \to \infty$.

(ii) Prove that the IVP (2) for $u_0 \in C(\overline{\Omega})$ with $\inf_{x\in\Omega} u_0(x) \geq c > 0$ can not have a bounded classical solution $u(x,t)$ for sufficiently large T. Hint: comparison principle

> Classical solutions u, $v \in C_1^2(G) \cap C^0(\overline{G})$ of (2) with $u \leq v$ on $\Omega \times \{0\}$ and $\frac{\partial u}{\partial \nu} \leq \frac{\partial v}{\partial \nu}$ on $\partial\Omega \times [0,T)$ satisfy $u \leq v$ on $\overline{G}$.

On the method of characteristics. The method of characteristics associates to a nonlinear PDE of first order

$$F(Du,u,x) = 0 \quad \text{in } \Omega \subset \mathbb{R}^n\,, \tag{3}$$

a system of ordinary differential equations

$$\begin{cases} \dot p(s) &= -D_xF(p(s),z(s),x(s)) - D_zF(p(s),z(s),x(s))p(s)\,, \\ \dot z(s) &= D_pF(p(s),z(s),x(s)) \cdot p(s)\,, \\ \dot x(s) &= D_pF(p(s),z(s),x(s))\,, \end{cases} \tag{4}$$

with $F(p(s), z(s), x(s)) = 0$ for s in a suitable interval $I \subset \mathbb{R}$. A solution $u(x)$ of boundary value problem (BVP)

$$\begin{cases} F(Du, u, x) = 0 & \text{in } \Omega \subset \mathbb{R}^n\,, \\ u = f & \text{on } \Gamma \subseteq \partial\Omega\,, \end{cases} \tag{5}$$

with *non-characteristic* boundary conditions, can be constructed from solutions of (4) with suitable initial conditions, see Evans *'Partial Differential Equations'* Section 3.2.

Exercise 3. Consider the BVP for the *Eikonal*-equation in geometrical optics

$$\begin{cases} |\nabla u| = 1 & \text{in } \Omega = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}\,, \\ u = \frac{x}{\sqrt{2}} & \text{on } \partial\Omega\,. \end{cases} \tag{6}$$

Compute a solution of (6) via the method of characteristics.

Hint: In our setting, the initial conditions

$$x(0) = x^0 = \begin{pmatrix} x \\ 0 \end{pmatrix} \in \partial\Omega\,, \quad z(0) = z^0\,, \quad p(0) = p^0 = \begin{pmatrix} p_1^0 \\ p_2^0 \end{pmatrix} \in \mathbb{R}^2\,, \tag{7}$$

have to satisfy (the compatibility conditions),

$$z^0 = f(x^0)\,, \quad p_1^0 = \frac{\partial f}{\partial x}(x^0)\,, \quad F(p^0, z^0, x^0) = 0\,, \tag{8}$$

which are *non-characteristic*, if

$$D_{p_2} F(p^0, z^0, x^0) \neq 0\,. \tag{9}$$

Remark. This method of characteristics is local: it provides a solution of (6) in a neighborhood of a point $x(0) \in \partial\Omega$ in $\overline{\Omega}$ such that (9) holds.

Remark. Another method to find solutions of (6) would consist in applying Theorem 5.1 in the book of P.L. Lions:
P.L. Lions, *Generalized solutions of Hamilton-Jacobi equations*, Volume 62, London Pitman, 1982.

Exercise 4. Consider the Cauchy problem for a *viscous* HAMILTON-JACOBI-*equation*

$$\begin{cases} u_t - \varepsilon\Delta u + b|\nabla u|^2 = 0 & \text{for } (x, t) \in \mathbb{R}^n \times (0, \infty)\,, \\ u(x, 0) = g(x) & \text{for } x \in \mathbb{R}^n\,, \end{cases} \tag{10}$$

with $b \in \mathbb{R}$, $\varepsilon > 0$ and a function $g : \mathbb{R} \to \mathbb{R}$.

(i) Suppose u is a classical solution of (10). Compute a function $\phi : \mathbb{R} \to \mathbb{R}$, such that $v = \phi(u)$ is a solution of a linear PDE.

(ii) Determine the Cauchy problem for $v = \phi(u)$.

Solutions will be discussed on Thursday 14th of March 2017.