Vienna University of Technology SS 2019
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Exercise Sheet 10 "Nonlinear Partial Differential Equations"
(parabolic PDEs)

Exercise 1. Let $n = 2$, $T > 0$ and $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain. Let $X := L^2([0,T],V)$, $V := \{\psi \in H_0^1(\Omega),\ \mathrm{div}(\psi) = 0 \text{ a.e. in } \Omega\}$, and H be the closure of V in $L^2(\Omega)$. Consider the Navier-Stokes equation

$$\partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p = f \text{ in } \Omega \times [0,T],$$

with

$$\mathrm{div}\, u = 0 \text{ in } \Omega \times [0,T],\ u = f \text{ on } \partial\Omega \times [0,T], \text{ and } u(.,0) = u_0 \text{ in } \Omega,$$

where $f \in L^2([0,T],V')$.

(i) Recall the definition of a weak solution of the above Navier-Stokes equation as well as the Gagliardo-Nirenberg inequality.

(ii) Complete the proofs of Theorems 3.25 and 3.26 in the lecture notes by proving the two following points left there as exercises:

a. Concerning uniqueness of the weak solution (Theorem 3.25). Show that, if u_1 and u_2 are two weak solutions of the transiente Navier-Stokes equation, it holds:

$$\int_\Omega (u_2 \cdot \nabla)w \cdot w\, dx = 0,$$

where we recall $w := u_1 - u_2$.

b. Concerning the properties of the differential operator $A(\phi) := -\Delta\phi + (\phi \cdot \nabla)\phi$ used in the proof of Theorem 3.26. Show that for $T > 0$,

$$A : X \cap L^\infty([0,T],H) \to X \text{ is bounded.}$$

Where do we use this property of A in the proof of Theorem 3.26?

Remark (a bit of history). In is PhD thesis in 1933, Jean Leray proved the existence of weak solutions on $\mathbb{R}^+$ (also referred as Leray's solutions in the literature) when $n = 2, 3$ of the Navier-Stokes equation. Notice that at this period, Sobolev spaces had not been introduced yet, he worked with a dual definition of what we now call $H^1(\Omega)$. He then proved the stability of weak solutions when $n = 2$ leading to the uniqueness of a weak solution. When $n = 3$, assuming a little bit more regularity of a weak solution on an interval $[0,T]$ (actually $L^4([0,T],V)$), he proved uniqueness on $[0,T]$ of a weak solution. Finally, still when $n = 3$, criterions were given to ensure that weak solutions are $L^4([0,T],V)$ for times T not too large if the data (u_0, f) are smooth enough. Let us mention that lower bounds when $n = 3$ on the maximal existence time T^* of a weak solution in $L^4_{loc}([0,T^*),V)$ were also provided in the literature. Global smoothness and uniqueness of a weak solution when $n = 3$ are still a very active field of research. Source: https://www.ljll.math.upmc.fr/chemin/pdf/2016M2EvolutionW.pdf

Exercise 2. Let $n = 2, 3$ and $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain. Let $u_0 \in L^2(\Omega)$. Consider the initial/boundary-value problem for the incompressible Navier-Stokes equation:

$$\partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p = 0 \text{ in } \Omega \times \mathbb{R}_+^*,$$

with

$$\operatorname{div} u = 0 \text{ in } \Omega \times \mathbb{R}_+^*, \ u = 0 \text{ on } \partial\Omega \times \mathbb{R}_+, \text{ and } u(.,0) = u_0 \text{ in } \Omega.$$

Show that a classical solution (u, p) satisfies the estimate

$$\text{for all } t \geq 0, \ \|u(.,t)\|_{L^2(\Omega)} \leq e^{-\lambda t}\|u_0\|_{L^2(\Omega)}$$

for some $\lambda > 0$ independent of t and (u, p).

Exercise 3. Let $C > 0$. Consider for each $m > 1$ the function

$$U_m : \mathbb{R}^n \times \mathbb{R}_+^* \to \mathbb{R}, \ (x,t) \mapsto t^{-\lambda}\Big(C - K\frac{|x|^2}{t^{\frac{2\lambda}{n}}}\Big)_+^{\frac{1}{m-1}},$$

where $\lambda := \frac{n}{n(m-1)+2}$ and $K := \frac{\lambda(m-1)}{2mn}$. Let us mention that the function U_m is known as Barenblatt-solution of the porous medium-equation

$$\partial_t u - \Delta(u^m) = 0 \text{ in } \mathbb{R}^n \times \mathbb{R}_+^*.$$

Part 1.
(i) Show that $t > 0 \mapsto \int_{\mathbb{R}^n} U_m(x,t)\,dx$ is constant and $U_m \to M\delta$ where δ is the Dirac distribution and $M := \int_{\mathbb{R}^n} U_m(x,t)\,dx$.
(ii) Let us assume that U_m is normalized such that $M = 1$: what do you expect to be the limit behaviour of U_m when $m \to 1^+$?
Prove that your guess is true. You can use without proof that $M = 1$,

$$C = 1/D^{\frac{1}{\gamma}},$$

where $D = \frac{1}{2}K^{-\frac{n}{2}}n \times \omega_n\, B(\frac{n}{2}, \frac{m}{m-1})$ and $\gamma = \frac{n}{2(m-1)\lambda}$, w_n being the volume of the unit ball in $\mathbb{R}^n$ and B the Euler beta function.

Part 2.
(i) Discuss the regularity of U_m.
(ii) Show that, for $\tau > 0$, the function $(x,t) \mapsto U_m(x, t+\tau)$ is a weak solution of the porous medium-equation

$$\partial_t u - \Delta(u^m) = 0 \text{ in } \mathbb{R}^n \times \mathbb{R}_+^*.$$

Solutions will be discussed on Thursday 6th of June 2019.