

Übungsaufgaben zur VU Computermathematik Serie 8

In all examples we use the package `LinearAlgebra`. Vectors u are generally to be understood as columns vectors, and u^T is the corresponding row vector. Then, $u^T v$ is the dot product (inner product) $u \cdot v$, and $u v^T$ is the outer product (or dyadic product), which is a matrix.

Choose appropriate test data to verify your solutions. Consult the help pages and be aware again that learning by doing is an essential feature in the exercises on computer mathematics.

Exercise 8.1. Let \mathcal{U} be a linear subspace of \mathbb{R}^3 of dimension 2 (i.e., a plane containing 0). We wish to determine the matrix representation of the projector P which projects $x \in \mathbb{R}^3$ onto \mathcal{U} in direction of a given vector $0 \neq w \notin \mathcal{U}$. P is uniquely determined by the requirements

$$P u = u, \quad P v = v, \quad P w = 0,$$

where $u, v \in \mathcal{U}$ are any linearly independent vectors spanning \mathcal{U} .

Design a Maple procedure

```
screw_projector(u::Vector, v::Vector, w::Vector)
```

which returns the matrix P in form of an object of type `Matrix`. Use `LinearSolve`. What happens if $w \in \mathcal{U}$ or if u, v are linearly dependent?

Remark: If $w \perp \mathcal{U}$, the outcome is the orthogonal projector onto \mathcal{U} .

Exercise 8.2. A differentiable vector field $S: \mathbb{R}^2 \supset G \rightarrow \mathbb{R}^3$ defines a parametrization of a smooth surface in \mathbb{R}^3 . The points on the surface are given by $(x, y, z) = S(u, v) = (S_1(u, v), S_2(u, v), S_3(u, v))$, where $(u, v) \in G \subset \mathbb{R}^2$.

Let $G = [a, b] \times [c, d]$ be a rectangle in (u, v) -plane. Then, the area of the surface is determined by the double integral

$$\int_F dS := \int_{v=c}^d \int_{u=a}^b \sqrt{m(u, v)} \, du \, dv, \quad \text{with } m(u, v) := \det \begin{pmatrix} \frac{\partial S}{\partial u} \cdot \frac{\partial S}{\partial u} & \frac{\partial S}{\partial u} \cdot \frac{\partial S}{\partial v} \\ \frac{\partial S}{\partial u} \cdot \frac{\partial S}{\partial v} & \frac{\partial S}{\partial v} \cdot \frac{\partial S}{\partial v} \end{pmatrix}.$$

Here,

$$\frac{\partial S}{\partial u} = \begin{pmatrix} \frac{\partial S_1}{\partial u} \\ \frac{\partial S_2}{\partial u} \\ \frac{\partial S_3}{\partial u} \end{pmatrix}$$

(analogously for v), und \cdot is the dot product.

Use this formula in Maple to compute the area of the unit sphere in \mathbb{R}^3 . In spherical coordinate, the unit sphere is specified as

$$S(u, v) = \begin{pmatrix} \sin u \cos v \\ \sin u \sin v \\ \cos u \end{pmatrix}, \quad (u, v) \in [0, \pi] \times [0, 2\pi].$$

Exercise 8.3. Let $A \in \mathbb{R}^{n \times n}$ be invertible, and $U, V \in \mathbb{R}^{n \times k}$. Then, the *Sherman-Morrison-Woodbury*-formula holds: $A + UV^T \in \mathbb{R}^{n \times n}$ is invertible iff $I + V^T A^{-1} U \in \mathbb{R}^{k \times k}$ is invertible, and

$$(A + UV^T)^{-1} = A^{-1} - A^{-1} U (I + V^T A^{-1} U)^{-1} V^T A^{-1}.$$

This formula can be used to compute the inverse $(A + UV^T)^{-1}$, assuming A^{-1} is already known. The additional effort involves only the inverse $(I + V^T A^{-1}U)^{-1} \in \mathbb{R}^{k \times k}$, and using the SMW formula is more efficient than direct inversion of $(I + V^T A^{-1}U)^{-1}$ if $k \ll n$.

Implement this formula in form of a procedure

```
SMW_Inverse(AI::Matrix,U::{Matrix,Vector[column]},V::{Matrix,Vector[column]})
```

In AI, the given inverse A^{-1} is passed. For the case $k = 1$, admit that U,V are specified in form of objects of type `Vector[column]` instead of `Matrix`.

In practical applications, such a formula is evaluated in floating point mode using `evalf`.

Exercise 8.4. A *data sparse* representation of a linear mapping or of its coefficient matrix A is a procedural representation of the mapping which does not involve the explicit form of the matrix. Example:

$$Ax = \prod_{k=1}^m (I - u_k v_k^T) \cdot x = (I - u_m v_m^T) \cdots (I - u_1 v_1^T) \cdot x, \quad u_k, v_k \in \mathbb{R}^n.$$

Here, the vectors $u_k, v_k \in \mathbb{R}^{n \times n}$ contain the full information about the mapping.

Design a procedure `mvmul(U::Matrix,V::Matrix,x::Vector)` which computes Ax without building the matrix A . (Here, the columns of $U, V \in \mathbb{R}^{n \times k}$ represent the vectors $u_k, v_k, k = 1 \dots m$.) Use inner products and scalar · vector multiplications only.

Hint: Since matrix multiplication is associative, we have $(uv^T)x = u(v^T x) = (v^T x)u$.

Exercise 8.5. The function `LinearAlgebra[CharacteristicPolynomial]` generates the characteristic polynomial $\pi(z) := \det(zI - A)$ of a given square matrix $A \in \mathbb{C}^{n \times n}$ ($I =$ unit matrix).

Use `CharacteristicPolynomial` to verify the *Cayley-Hamilton theorem*

$$\pi(A) = 0 \quad (\text{zero matrix})$$

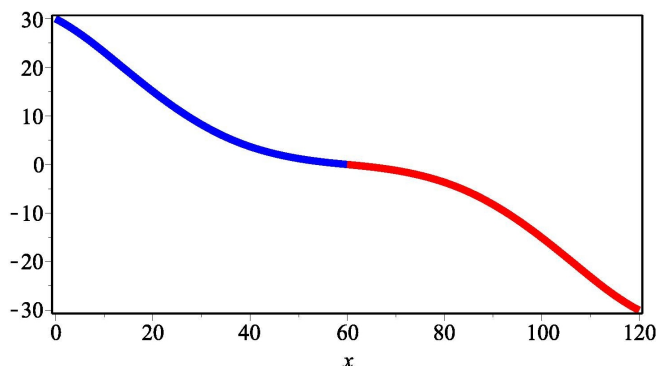
by means of examples. Here, $\pi(A)$ is the corresponding matrix polynomial.

Design a procedure `CHsolve(A::Matrix,b::Vector)` which computes the solution $x = A^{-1}b$ of a linear system $Ax = b$ exploiting the assertion of the Cayley-Hamilton theorem. Here A is assumed to be invertible. What happens if A is not invertible?

Hint: See Exercise 3.3.

Remark: This is only reasonable for small dimension n . The assertion of the Cayley-Hamilton theorem is mainly of theoretical interest.

Exercise 8.6.



We design the profile of a ski-jump. To this end we represent the profile of its *inrun* by a polynomial $p(x)$ of degree 5:

- Start at top left corner, $(x_0, y_0) = (0, 30)$
- End of jump-off platform at $(x_1, y_1) = (60, 0)$.

Furthermore we require:

- $p'(x_0) = -\frac{1}{2}$
- $p'(x_1) = -1/10$
- $p''(x_1) = 0$,
- and
- $p(120) = -30$,

such that the polynomial also approximates a possible **flight trajectory** for $x > 60$.

Compute the coefficients of this polynomial in Maple by setting up the equations and solving the corresponding linear system. Furthermore, determine the turning points of the profile, and reproduce the above plot.

Exercise 8.7. Design a procedure `decomp(A::Matrix)` which expects an rectangular matrix $A \in \mathbb{R}^{m \times n}$ with rational coefficients as an argument and which returns bases of the linear subspaces

$$\text{image}(A) := \{Ax : x \in \mathbb{R}^n\}$$

and

$$\text{kernel}(A^T) := \{y \in \mathbb{R}^m : A^T y = 0\}$$

of \mathbb{R}^m . The subspace $\text{image}(A)$ is also called the column space of A (i.e., the subspace of \mathbb{R}^m spanned by the columns of A), and $\text{kernel}(A)$ is the kernel, or nullspace, of A^T . These bases can be computed using elimination. In the `LinearAlgebra` package there are functions which directly support the determination of these subspaces – find out using the help page for this package. Your procedure should return the list `[ImA, KerAt]`, where `ImA` and `KerAt` are matrices containing the respective basis vectors as their columns. The case of a zero subspace should be correctly handled! Test several examples of different types.

Remark: It is an elementary fact that these subspaces give an orthogonal subspace decomposition

$$\mathbb{R}^m = \text{image}(A) \oplus \text{kernel}(A^T), \quad \text{with } \text{image}(A) \perp \text{kernel}(A^T).$$

However, the bases delivered by Maple are themselves not orthogonal in general.

Exercise 8.8. Let $p(z)$ be a polynomial of degree n with real coefficients. We write it in the form

$$p(z) = z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n, \quad a_i \in \mathbb{R}.$$

The classical *Routh-Hurwitz* theorem says that

$$\text{Re } z_j < 0 \text{ for all zeros } z_j \text{ of } p(z) \iff H_k > 0 \text{ for all } k = 1 \dots n,$$

where the H_k are the leading minors (subdeterminants) of the matrix H defined by the following pattern:

$$H = \begin{pmatrix} a_1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ a_3 & a_2 & a_1 & 1 & 0 & 0 & 0 & \dots & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 & 0 & \dots & 0 \\ a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{2n-1} & a_{2n-2} & a_{2n-3} & a_{2n-4} & a_{2n-5} & a_{2n-6} & a_{2n-7} & \dots & a_n \end{pmatrix} \in \mathbb{R}^{n \times n},$$

where we set $a_k := 0$ for $k > n$.

Design a procedure `RouthHurwitz(...)`, which implements this test for polynomials with integer coefficients. To compute the subdeterminants, use `LinearAlgebra[Determinant]`. Represent the polynomial coefficients by a `Vector`, and return `true` or `false`, respectively.