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Übungsaufgaben zur VU Computermathematik

Serie 6

Einige plot-Befehle und einige weitere Befehle (z.B. `coeff`*,* `degree`*,* `piecewise`*) wurden in der VO noch nicht genauer besprochen. Dafür werden jeweils Hinweise gegeben. Konsultieren Sie für genauere Details die Maple-Hilfe.*

Exercise 6.1. *Parametric plots.*

a) The *Möbius strip* is a two-dimensional manifold in $\mathbb{R}^3$ which is not orientable. A parametric representation is given by the coordinate functions

$$\begin{aligned} x(\varphi,t) &= \cos\varphi\,(r + t\cos(\varphi/2)) \\ y(\varphi,t) &= \sin\varphi\,(r + t\cos(\varphi/2)) \qquad (\varphi \in [0,2\pi],\ t \in [-r,r]) \\ z(\varphi,t) &= t\,\sin(\varphi/2)) \end{aligned}$$

Here, $2\,r$ is the width of the strip (for a given numerical value $r > 0$).

Use `plot3d` and play with plot parameters in order to produce a nice plot:

```
plot3d([x(phi),t),y(phi,t),z(phi,t)],phi=0..2*Pi,t=-r..r,...);
```

b) Given a curve in the plane in parametric form, e.g., the *Archimedian spiral*

$$\begin{aligned} x(\varphi) &= \varphi\,\cos\varphi \\ y(\varphi) &= \varphi\,\sin\varphi \end{aligned} \qquad (\varphi \geq 0)$$

you may expect that this can be plotted in analogy to **a)** using `plot` in the form

```
plot([x(phi),y(phi)],phi=0..phimax,...);
```

(with a chosen value for `phimax`). Try out − what happens? Consult the help page for `plot` to check how to realize such a two-dimensional *parametric plot.* Play with plot parameters in order to produce a nice plot of the spiral and of some other curve of your choice.

Hint: Note that you also can manipulate a plot interactively in various ways using the right mouse button or using the context menu shown on top when you click into the plot window.

Exercise 6.2. *Manipulating polynomials. Complex arithmetic and harmonic polynomials.*

a) Design a function `polcoe(p,z)` which expects a polynomial expression

$$c_0 + c_1\,z + c_2\,z^2 + \ldots c_n\,z^n$$

in the variable `z` as arguments and returns the list of coefficients $[c_0, c_1, \ldots, c_n]$.

Hint: Use `degree` and `coeff`.

b) Design a function for the reverse operation.

c) Choose an arbitrary polynomial $p(z)$ with real or complex coefficients, e.g. $p(z) = (3+i)\,z^3 - i\,z^2 + 5\,z - 1 - i$. Then, using

```
x,y:='x','y': assume(x,real): assume(y,real):
```

declare the variables `x,y` to represent *real* values. Now evaluate `p(x+I*y)` using `evalc`, and extract real and imaginary parts using `Re` and `Im`. This yields two bivariate polynomials $u(x,y)$ and $v(x,y)$ with real coefficients in the variables x and y. Now, evaluate (possibly using `simplify`)

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} \qquad \text{and} \qquad \frac{d^2v}{dx^2} + \frac{d^2v}{dy^2}$$

What do you observe?

Exercise 6.3. *Smoothness analysis for a real function.*

Consider the function $f\colon\ \mathbb{R}\to\mathbb{R}$, defined by

$$f(x)=\sqrt{1+x^2+\sqrt{x^4+\sqrt{x^6}}}$$

a) Use Maple to check whether f' and f'' are well-defined and continuous at $x=0$, possibly in the sense of continuous extension (*hebbare Unstetigkeit*).

Hint: Use `limit`.

b) Generate a plot showing the graphs of f, f', and f'' over an interval containing 0. Play with plot-parameters in order to produce a nice-looking plot.

Exercise 6.4. *Some elementary calculus.*

a) Use Maple to perform a complete analysis (*Kurvendiskussion*) for the real function

$$f(x)=\arctan(x^3)$$

including a plot. Also, use `pointplot` and `display` from the `plots` package to add markers for special points (zeros, local minima, local maxima, turning points) to the plot. Use different colors.

b) Same as **a)**, for the real function

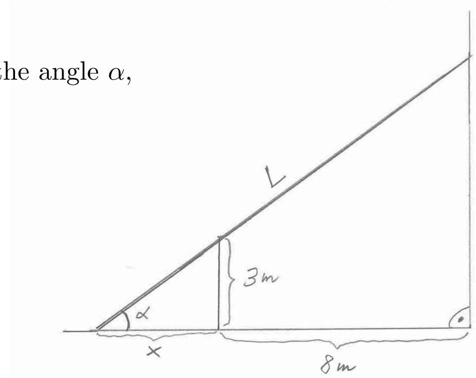
$$f(x)=\frac{(\ln x)^2}{x}$$

Exercise 6.5. *An extremal value problem.*
(See figure:) A rod is positioned as shown in the figure. Its length L depends on the angle α, with $L(\alpha)\to\infty$ for $\alpha\downarrow 0$ and for $\alpha\uparrow\frac{\pi}{2}$.

For what angle $\alpha\in(0,\frac{\pi}{2})$ does the length $L=L(\alpha)$ become minimal?
Determine the minimal length and the corresponding distance x (see figure).

Solve this problem with the help of Maple.
At the end, evaluate the results in floating point using `evalf`.
Also, give the resulting value for α in degrees $^\circ$.

Remark: The minimal length is *not* given by $L=15\,\mathrm{m}$.



Exercise 6.6. *Functions defined in a piecewise way.*

Let a function $f(x)$ defined in a piecewise manner, e.g.

$$f(x):=\begin{cases}-1, & x<0\\ 0, & x=0\\ 1+x, & x>0\end{cases}$$

a) Design a function which, using `if ... end if` or `` `if`(...) ``, implements this (or a similar) example.

Can you differentiate or integrate this function using `diff`,`D`, or `int`?

Can you plot it? If not, try `plot('f(x)',...)`

b) Alternative: Define the same function using the `piecewise` construct (look at the help page), and try again. Evaluate its indefinite integral and a definite integral.

c) Assume that `X` is a strictly monotonically ordered list of numbers representing points on the real line, and let `Y` represent a list of function values, with `nops(Y)=nops(X)-1`. Design a procedure `generate_stepfunction(X,Y)` which uses the `piecewise` construct to return the corresponding step function (*Treppenfunktion*)

$$f(x):=\begin{cases}0, & x<x_1 \quad (\text{i.e., for } x\in(-\infty,x_1))\\ y_1, & x<x_2 \quad (\text{i.e., for } x\in[x_1,x_2))\\ y_2, & x<x_3 \quad (\text{i.e., for } x\in[x_2,x_3))\\ \vdots & \vdots\\ y_{n-1}, & x<x_n \quad (\text{i.e., for } x\in[x_{n-1},x_n))\\ 0, & \text{otherwise} \quad (\text{i.e., for } x\ge x_n)\end{cases} \qquad (n=\texttt{nops(X)})$$

Choose an example and plot the resulting step function.

Hint: Use `seq` to generate the `piecewise` construct.

Exercise 6.7. *Error analysis of some numerical integration methods.*

If a definite integral

$$J(f) := \int_a^b f(x)\,dx$$

cannot be exactly determined, numerical approximation methods are applied. [1] We consider simple examples of such methods which only require evaluation of some function values. For simplicity, we consider the case $a = 0$ and $b = h$; think of h as the length of a small interval. In the following, f is any function (not explicitly specified).

a) A simple integration scheme is the *trapezodial rule* (with an obvious geometric interpretation)

$$T(f) := \frac{h}{2}\big(f(0) + f(h)\big) \approx J(f) = \int_0^h f(x)\,dx$$

Use `int` and `taylor` to compute the Taylor expansion of the error $T(f) - J(f)$ with respect to the variable h at 0. What is the leading power of h in this expansion? (This is called the *order* of the approximation.) In what way does the leading error term depend on the integrand f?

b) Now we construct a better approximation method. Use the ansatz

$$S(f) := h\,\big(a\,f(0) + b\,f(\tfrac{h}{2}) + a\,f(h)\big)$$

and again use `taylor` to compute the Taylor expansion of the error $S(f) - J(f)$. Can you find coefficients a and b such that the leading power of h in this expansion is 5 ? In what way does the resulting leading error term depend on the integrand f ?

c) Test both numerical methods on a simple example (e.g., $f = \exp$, $h = 0.1$) and compare the results. Use `evalf`.

Exercise 6.8. *Two small procedures.*

a) Design a procedure `checktype(A,t)` which does the same job as the function `checktype` from **5.4 d)**.

Hint: Use logical operations and `evalb` within a `for` loop.

b) Design a procedure `ismonotone(L)` which checks if a list `L` with numerical entries is [strictly] monotonously decreasing or increasing. Return

- -2 if the list is strictly monotonously decreasing,
- -1 if the list is monotonously decreasing but not strictly,
- $+1$ if the list is monotonously increasing but not strictly,
- $+2$ if the list is strictly monotonously increasing,
- *but:* 0 if all entries in the list are identical (which would be compatible with -1 as well as $+1$),
- `NULL` otherwise.

[1] Remark: Some advanced numerical integration methods are implemented in Maple/`int`; they are activated if `int` is called with the option `numeric`.