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Übungsaufgaben zur VU Computermathematik

Serie 8

In all examples we use the package `LinearAlgebra` and the data types `Vector` and `Matrix`. Some numerical aspects are included.

Exercise 8.1: ***Playing with Hilbert matrices.***

The symmetric matrix

$$H_n = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \left(\frac{1}{i+j-1} \right)_{i,j=1\ldots n} \in \mathbb{R}^{n\times n}$$

is called Hilbert matrix (see `LinearAlgebra[HilbertMatrix]`).

a) The inverse of a Hilbert matrix has integer entries which become large in size very quickly with increasing dimension n.

To illustrate this, produce a `pointplot` *of the numbers*

$$\max_{1\le i,j\le n} |(H_n^{-1})_{ij}|\,.$$

for $n = 2, 3, \ldots$. *(To see the trend, it may be more useful to plot logarithms.)*

b) *Use* `plots[matrixplot]` *to visualize* H_n *and* H_n^{-1}, *e.g., for* $n = 10$ *or* $n = 20$.

c) *Use* `time()` *to measure the CPU time required for inversion (in exact rational arithmetic) of the Hilbert matrices* H_n *in dependence of the dimension* n. *A reasonable choice will be* $n = 10, 20, \ldots, 100$; *maybe larger. Produce a [logarithmic]* `pointplot` *of these numbers. What do you observe?*

d) This is just as an example for using an indexing function:

Define the matrix

$$H_{n,m} = \begin{pmatrix} 0 & \frac{1}{m+1} & \frac{1}{m+2} & \frac{1}{m+3} & \cdots \\ \frac{1}{m^2+1} & 0 & \frac{1}{m+3} & \frac{1}{m+4} & \cdots \\ \frac{1}{m^2+2} & \frac{1}{m^2+3} & 0 & \frac{1}{m+5} & \cdots \\ \frac{1}{m^2+3} & \frac{1}{m^2+4} & \frac{1}{m^2+5} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in \mathbb{R}^{n\times n}$$

For this purpose, use a function or a procedure depending on the two parameters $m > 1$ *and* $n \in \mathbb{N}$, *which generates such a matrix.*

Exercise 8.2: ***Investigation of a parameter-dependent matrix.***

Consider the matrix

$$A = \begin{pmatrix} c & c & 1 & 0 & c \\ 1 & 0 & 0 & c & 0 \\ 0 & 1 & 1 & 0 & 1 \\ c & 1 & 0 & 1 & 0 \\ 1 & c & 1 & 0 & c \end{pmatrix}$$

depending on a real parameter c. Use `LinearAlgebra`:

a) *For which values c is A invertible? Determine the inverse of A.*

b) *Same question as in* **a)**, *for the symmetric part $(A + A^T)/2$ instead of A.*

c) *Same question as in* **a)**, *for the skew-symmetric part $(A - A^T)/2$ instead of A.*

d) *For those values of c where A is not invertible, compute a representation for the kernel of A. What is the rank of A in these cases?*

e) *Same question as in* **b)**, *but for the 'real part' $(A + A^*)/2$, where c is admitted to be complex and A^* is the Hermitian transpose of A. Can you answer this question?*

If you fail, represent c in the form $c = a + i\,b$, $a, b \in \mathbb{R}$, and try again.

Hint: Use `HermitianTranspose`; then, c will automatically be interpreted as a complex number.

Exercise 8.3: ***A numerical study: Is this matrix indeed invertible?***

a) We consider a matrix A with floating point entries,

$$A = \begin{pmatrix} 1.11 & 4.44 \\ 3.33 & 13.30 \end{pmatrix}$$

Compute the inverse of A.

b) Assume that A is not given exactly (due to measurement or rounding errors); the original matrix may be the following, very similar one:

$$B = \begin{pmatrix} 1.111 & 4.440 \\ 3.333 & 13.320 \end{pmatrix}$$

Compute the inverse of B.

c) How can we verify that A is close to a singular matrix? This is a topic in courses on numerical linear algebra. Here, we simply look at the columns of A:

Compute the angle between these columns.

As you see, the columns are almost parallel. If they are exactly parallel, the matrix becomes singular (this is the case for B). Therefore we say: A is 'almost singular'.

(Remark: When you try to solve such a linear system $A\,x = b$, then the answer will depend in a very sensitive way on perturbations in its coefficients. Such a system is called ill-conditioned.)

Exercise 8.4: ***Hilbert matrices are ill-conditioned. Computing eigenvalues.***

Hilbert matrices are increasingly ill-conditioned with increasing dimension, i.e., they get 'closer and closer' to a singular matrix. We investigate this property with the following experiment, where we consider H_n perturbed by a multiple of the identity matrix.

a) *Consider the matrices $A_n := H_n - \varepsilon\, I_n$ (H_n from* **8.1**, *I_n = identity matrix) for $n = 2, 3, 4, 5, \ldots$, and find the smallest (in size) $\varepsilon \in \mathbb{C}$ such that A_n is a singular matrix.*

Hint: Use `Determinant` (or `CharacteristicPolynomial`), `solve`, and `evalf`.

b) For a square matrix A, a real or complex number ε such that $A - \varepsilon\, I$ is singular is called an eigenvalue of A.

Repeat the computation from **a)** *using* `Eigenvalues(...)` *and* `Eigenvalues(evalf(...))`.

(The latter computes eigenvalues by a more efficient numerical algorithm, and you can also test larger dimensions, e.g., $n = 10$, $n = 20$.)

Remark: In linear algebra it is shown that all eigenvalues of a symmetric matrix like H_n are real.

Exercise 8.5: *Constructing a projector.*

Let $\mathcal{U}$ be a linear subspace of $\mathbb{R}^3$ of dimension 2 (i.e., a plane containing 0). We wish to determine the matrix representation of the projector P which projects $x \in \mathbb{R}^3$ onto $\mathcal{U}$ along the direction of a given vector $0 \neq w \notin \mathcal{U}$. P is uniquely determined by the requirements (make a sketch)

$$P\,u = u, \quad P\,v = v, \quad P\,w = 0,$$

where $u, v \in \mathcal{U}$ are any linearly independent vectors spanning $\mathcal{U}$.

a) *Design a procedure* `projector(u::Vector,v::Vector,w::Vector)` *which returns the matrix P in form of an object of type* `Matrix`.

Hint: Use `LinearSolve` to solve the corresponding matrix equation. What happens if $w \in \mathcal{U}$ or if u, v are linearly dependent?

b) *What is the rank of P? – verify.*

c) *What is P^2, and why?*

Remark: If $w \perp \mathcal{U}$ then the outcome is the orthogonal projector onto $\mathcal{U}$. If, on the other hand, w is almost parallel to $\mathcal{U}$, then the action of the projector is very sensitive to data perturbations ('schleifender Schnitt').

Exercise 8.6: *Evaluating a matrix polynomial.*

Let $p := x \mapsto c_0 + c_1\, x + \ldots + c_k\, x^k$ be a polynomial function. Then, the matrix-valued function

$$A \;\mapsto\; p(A) := c_0\, I_n + c_1\, A + \ldots + c_k\, A^k, \quad \text{with } A \in \mathbb{R}^{n\times n} \text{ (or } \mathbb{C}^{n\times n}\text{)},$$

is called a matrix polynomial.

- *Design a procedure* `peval(p,A,v)` *which computes the matrix-vector product $p(A)\cdot v$ for given p, a given square matrix A, and a given vector v.*

— *Include a check for compatible dimensions concerning* `A::Matrix` *and* `v::Vector`.

— *Use efficient evaluation using a Horner-like-scheme,*

$$p(A)\cdot v \;=\; c_0\, v + A\left(c_1\, v + A\left(c_2\, v + \cdots\right)\right)$$

which uses only matrix-vector products but no matrix-matrix products.

Hint: `degree(...,x)` returns the degree of a polynomial expression $p(x)$.

Exercise 8.7: *Matrix representation of a linear mapping.*

Let a linear mapping $\psi\colon \mathbb{R}^n \to \mathbb{R}^m$ be given in form of a procedure `psi`.

- *Design another procedure which returns its coefficient matrix $A \in \mathbb{R}^{m\times n}$ with respect to the canonical bases in $\mathbb{R}^n$ and $\mathbb{R}^m$.*

Remark: We consider ψ to be 'black box', i.e., we only know that it represents a linear mapping. But we need to know what the dimensions m and n are. For this exercise you may simply assume that m and n are a priori known.

Exercise 8.8: ***Visualization of linear mappings.***

a) *With* `plots[arrow]` *you can draw arrows. Use this to visualize the behavior of a linear mapping* $\psi\colon \mathbb{R}^3 \to \mathbb{R}^3$ *represented by a matrix* $A \in \mathbb{R}^{3\times 3}$*, by drawing the parallelepiped spanned by the image of the unit vectors* $(1,0,0)$, $(0,1,0)$ *and* $(0,0,1)$ *under the mapping (these are the columns of* A*). Choose an example and produce a nice plot.*

(You may first work on the two-dimensional case.)

b) $(*)$ Another visualization is provided by the image of the unit sphere under the mapping. To this end we use spherical coordinates,

$$x = \cos\theta\,\cos\phi,$$
$$y = \cos\theta\,\sin\phi,$$
$$z = \sin\theta,$$

with $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\phi \in [-\pi, \pi]$, and use `plot3d`.

Produce a nice plot. Also use `display` *to combine this with a plot of the unit sphere. Use different colors and set the option* `transparency=0.5` *or similar.*

Hint: With `convert(...,list)` you can convert a `Vector` into a list. For plotting, observe the role of the parameter `scaling,`, which is relevant here.