Winfried Auzinger
Dirk Praetorius

Sommersemester 2018
2. Mai 2018

Übungsaufgaben zur VU Computermathematik

Serie 6

Exercise 6.1: ***Limits and infinite series.***

a) *Compute the following limits (these are elementary examples):*

$$\lim_{n\to\infty}\Big(\sqrt{n+\sqrt{n}}-\sqrt{n}\Big) \qquad \lim_{x\to 0}\frac{e^x-1-x-x^2-x^3-x^4}{x^5} \qquad \lim_{x\to 0}\frac{e^x-1-x-\dfrac{x^2}{2}-\dfrac{x^3}{6}-\dfrac{x^4}{24}}{x^5}$$

$$\lim_{n\to\infty}\Big(1+\frac{\mathrm{i}}{n}\Big)^n \ \ (\mathrm{i} = \text{imaginary unit}) \qquad \lim_{x\to 0}\frac{\ln(1+\ln(1+x))}{x} \qquad \lim_{x\to 1}\frac{\ln x}{\sin(\pi x)}$$

b) *Compute the values of the following infinite series (not elementary). Also evaluate them using* `evalf`*:*

$$\sum_{k=1}^{\infty}\frac{1}{k^4} \qquad \sum_{k=1}^{\infty}\frac{1}{k^8} \qquad \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^4} \qquad \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^8}$$

c) Again playing with π a little bit:

Series of the form $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^p}$ converge faster and faster with increasing p (towards values closer and closer to 1).

For instance, you may take the sums

$$\sum_{k=1}^{n}\frac{1}{k^p}, \quad p = 4 \ \text{ or } \ 8$$

as reasonably fast converging approximations of their limit for $n\to\infty$.

Accepting numerical evaluation of square roots,[10] *use this to determine the minimal value of n such that 10 correct decimal digits of π are generated. Compare $p=4$ with $p=8$ using* **b)***; also try $p = 16$.*

(Set `Digits` to 20 and test using `evalf`.)

Exercise 6.2: ***Analyzing a real function.***

*Use Maple as a computational tool for analyzing the real function (*Kurvendiskussion*)*

$$f(x) = \arctan(x^5)$$

including nice plots of the function and its first and second derivatives.

Exercise 6.3: ***Integration of rational functions.***

a) *Compute the indefinite integrals of the real functions*

$$\text{(i)}\ \ f(x) = \frac{1}{1+x^4} \qquad \text{(ii)}\ \ f(x) = \frac{1}{1+x^8}$$

b) *Verify that the result obtained in* **a)**, (i) *is indeed correct by differentiating it.* Note that simplifying the outcome is not straightforward; you can try `normal` and separately simplify or factor numerator and denominator. (Some trial and error is unavoidable here.)

---

[10] This means that it is not a purely rational aprpoximation.

c) (∗) Same question as in **b)**, for **a)**, (ii). Here you first have to interpret the answer delivered by `int`:[11]

```
(1/8)*(sum(ln(x-_R)/_R^7,_R = RootOf(_Z^8+1)))
```

You have seen `RootOf` expressions before, and with your knowledge on integration of rational functions via partial fraction decomposition it should be obvious that this is the correct answer, even if it is not given in explicit form.

Convert it into explicit form, using `solve` *applied to the equation* $x^8 + 1 = 0$*. Then, proceed as in* **b)***: Differentiate, and try to recover the original integrand.* (This is more laborious than for **a)**, (i).)

Remark: Note that most roots of the equation $x^8 + 1 = 0$ are complex, Therefore, in contrast to **a)**, (i) we are dealing here with a formally complex representation of the integral.

Exercise 6.4: ***Harmonic sums and natural logarithm.***

We consider the harmonic sums

$$S_n = \sum_{j=1}^{n} \frac{1}{j}\,.$$

a) *Use*[12] `plots[listplot]` *to plot the points* $(x_n, y_n) = \big(n, S_n - \ln(n)\big)$ *in the* (x, y)*-plane. What do you observe for larger values of* n*?*

Hint: Use a `for` loop to generate the S_n and store the values $S_n - \ln n$ (evaluated in floating point) in a list.

b) *Perform an experiment to find* $N \in \mathbb{N}$ *such that for* $n \geq N$ *the relative error* $(S_n - \ln n)/S_n$ *is below 10 %. What about 1 % accuracy?*

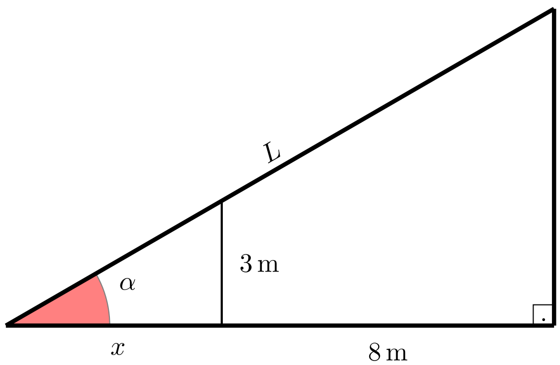
c) The limit $\lim_{n\to\infty} \big(S_n - \ln n\big)$ is called the Euler-Mascheroni constant, $\gamma \approx 0.5772156649$. *Perform an experiment to find* $N \in \mathbb{N}$ *such that for* $n \geq N$ *the relative error* $\big(S_n - (\ln n + \gamma)\big)/S_n$ *is below 1‰.*

(In Maple, the constant `gamma` is predefined, and you can evaluate it using *evalf*.)

d) *Determine experimentally the constant* $p > 0$ *such that* $S_n - (\ln n + \gamma) \sim n^{-p}$ *for* $n \to \infty$*.*

Hint: '$\sim$' means (approximate) equality up to a factor independent of n. Compare the results for $n = 10^2, 10^2, 10^3, \ldots$, then it becomes very plausible what the correct value of p must be.

Exercise 6.5: ***An extremum problem.***



A rod is positioned as shown in the figure. Its length L depends on the angle α, with $L(\alpha) \to \infty$ for $\alpha \downarrow 0$ and for $\alpha \uparrow \frac{\pi}{2}$.

Question: For which angle $\alpha \in (0, \frac{\pi}{2})$ does the length $L = L(\alpha)$ become minimal? The minimal length and the corresponding distance x is to be determined.

Solve this problem with the help of Maple, and evaluate the results in floating point using `evalf`*. Also, give the resulting value for* α *in degrees* °.

Remark: The minimal length is not given by $L = 15$ m.

[11] Here you see the answer delivered by Maple 2017.

[12] Syntax: `plots[listplot](...)` activates the function `listplot` contained in the package `plots`. You can also activate the complete package using `with(plots);` and simply call `listplot`.

Exercise 6.6: ***The discrete sine transform.***

Let $n \in \mathbb{N}$ and consider the points $t_j := j/(n+1)$, $j = 1 \dots n$. Then, $\Delta := (t_1, t_2, \dots, t_n)$ is a sequence of equispaced grid points in the interval $[0,1]$. The functions $s_k \colon \Delta \to \mathbb{R}$, $k = 1 \dots n$, defined by

$$s_k(t_j) = \sin(k\,\pi\,t_j), \quad j = 1 \dots n,$$

are discrete sine functions of varying frequency.

Let $x \colon \Delta \to \mathbb{R}$ be a function defined on Δ, and denote[13] $x_j := x(t_j)$, $j = 1 \dots n$. The discrete sine transform (DST) of x is defined as the vector of 'Fourier coefficients' $\hat{x} = (\hat{x}_1, \dots, \hat{x}_n) \in \mathbb{R}^n$,

$$\hat{x}_k = \sum_{j=1}^{n} x_j \, \sin(k\,\pi\,t_j), \quad k = 1 \dots n. \tag{DST}$$

The $\hat{x}_k$ are exactly the coefficients of a representation of x in the form

$$x = \frac{2}{n+1} \sum_{k=1}^{n} \hat{x}_k \, s_k\,,$$

i.e.,

$$x_j = \frac{2}{n+1} \sum_{k=1}^{n} \hat{x}_k \, \sin(k\,\pi\,t_j), \quad j = 1 \dots n. \tag{IDST}$$

This means that x is decomposed into discrete sine frequencies. (IDST) is called the inverse discrete sine transform; it is identical with the DST up to the scaling factor $\frac{2}{n+1}$.

a) *Design two procedures* `dst(y)` *and* `idst(y)` *which compute the* [I] DST *of* y *(we represent* $y = x$ *or* $y = \hat{x}$ *by a list). Verify by example that, indeed,* `idst(dst(x))=x` *and vice versa. Choose an example and use* `plots[pointplot]` *to visualize the behavior of the* $\hat{x}_k$ *in dependence of* k.

Natural choice for x: $x_j = f(t_j)$ for some given function $f \colon [0,1] \to \mathbb{R}$ satisfying $f(0) = f(1) = 0$. For smooth functions f the $\hat{x}_k$ should significantly decay with increasing k. For strongly oscillating functions the behavior is different (*test*).

b) *Verify by example that the rescaled* DST *(identical with the rescaled* IDST*)*

$$x^* := \mathrm{dst}^*(x) = \Big(\sqrt{\tfrac{2}{n+1}} \sum_{j=1}^{n} x_j \, \sin(k\,\pi\,t_j), \; k = 1 \dots n \Big)$$

is an isometric operation, i.e., $\sum_{k=1}^{n} (x_k^*)^2 = \sum_{k=1}^{n} x_k^2$.

Exercise 6.7: ***Newton type zero finders.***

Newton iteration is a standard approach for determining numerically (in floating point arithmetic) a zero ξ of a differentiable real function $f(x)$, starting from an initial guess x_0,[14]

$$x_i := x_{i-1} - \frac{f(x_{i-1})}{f'(x_{i-1})}, \quad i = 1, 2, 3, 4, \dots \tag{N1}$$

a) *Design a procedure* `newton(...)` *which expects a function* f, *an initial guess* $x_0 \in \mathbb{R}$ *and a tolerance parameter* tol *as its arguments and which performs the iteration* (N1) *until* $|f(x_i)| \le tol$. *Your procedure returns the list* $[i, x_i]$. *If the criterion* $|f(x_i)|$ *is not satisfied after* 100 *steps, we consider the iteration having failed. In this case, return* `[NULL]`. *(*`NULL` *is the Maple symbol for 'nothing'.)*

A simple test example: Approximation of $\sqrt{2}$: Solve the equation $x^2 = 2$ by Newton iteration, starting from $x_0 = 1.5$.

b) An improved iteration involving the second derivative f'' reads

$$x_i = x_{i-1} - \frac{f(x_{i-1})}{f'(x_{i-1})} \left(1 + \frac{1}{2} \, \frac{f(x_{i-1})\, f''(x_{i-1})}{f'(x_{i-1})^2} \right) \tag{N2}$$

Proceed as in **a)** *and compare both versions: Which one converges faster?*

Hint: For testing, choose e.g. `Digits=100` and tol = 1E-90.

[13] Think of a function which takes the value 0 for $t_0 = 0$ and $t_{n+1} = 1$. The $s_k(t_j)$ also have this property.

[14] (N1) means that x_{i+1} is the solution of the (at x_i) linearized problem $f(x_i) + f'(x_i)(x - x_i) = 0$.

Exercise 6.8: ***Asymptotic analysis of Newton-type zero finders.***

We study the local speed of convergence of the Newton iteration (N1) in a vicinity of an exact solution ξ in the following way: Assume we already know a good approximation $x = \xi + \varepsilon$ (with ε small). Now we consider a Newton step starting from x,

$$x \mapsto \hat{x} := x - \frac{f(x)}{f'(x)} \quad \text{with} \quad x = \xi + \varepsilon$$

and analyze the new error $\hat{x} - \xi$ in dependence of ε. *Proceed as follows:*[15]

a) *Generate the expression for $\hat{x}$ and use* `subs` *to replace x by $\xi + \varepsilon$. Now, use* `taylor` *to perform a Taylor expansion in ε about $\varepsilon = 0$ up to a remainder $O(\varepsilon^2)$. Simplify the resulting expression and use* `subs` *to replace the subexpression $f(\xi)$ by 0. Interpret the outcome.*

b) *Same as in* **a)***, for the modified Newton iteration* (N2)*. What do you see? Oops – use Taylor expansion of a higher order.*

---

[15] This exercise involves only symbolic operations – a short computer-aided proof. f is an 'anonymous' function, it is not defined in a particular way. `taylor` can work with such anonymous functions. Note that $f'(\xi)$ is not allowed to be 0, i.e., ξ must be a simple zero of f.