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Übungen zur Vorlesung
Computermathematik

Serie 11

Aufgabe 11.1. Use `\newenvironment` to generate a new *theorem*-environment. Write as well a *proof*-environment. The proof should start (as part of the environment) with bold-italic ***Proof.***. The end of the proof (as part of the environment) should be indicated with a right-aligned `\blacksquare` ■, i.e., there is a right-aligned ■ at the end of the proof. Formulate and prove the following theorem in LaTeX.

Theorem 1. *For $a, b \in \mathbb{R}$ and a continuous function $f : (a, b) \to \mathbb{R}$, the following two assertions are equivalent:*

(i) *f is uniformly continuous.*

(ii) *f has a continuous extension onto the compact interval $[a, b]$, i.e., there exists a function $\widehat{f} : [a, b] \to \mathbb{R}$ with $\widehat{f}(x) = f(x)$ for all $x \in (a, b)$.*

In this case the continuous extension $\widehat{f}$ is even unique.

Hint. The `itshape`-environment formats a text *italic*.
Surely, you have seen the implication (ii) $\Rightarrow$ (i) in your Analysis-1-lecture. For the converse implication (i) $\Rightarrow$ (ii), recall that uniformly continuous functions map Cauchy-sequences onto Cauchy-sequences.

Aufgabe 11.2. Extend Aufgabe 11.1. Formulate the following assertion as a theorem, prove it with techniques of linear algebra and write the theorem with its proof in LaTeX, where all appearing references should be realized via `\label` and `\ref` etc. If $A \in \mathbb{R}^{n\times n}$ is a matrix with $\sum_{j,k=1}^{n} x_j A_{jk} x_k > 0$ for all $x \in \mathbb{R}^n$, then A is regular.

Aufgabe 11.3. Write a LaTeX-file in which the following theorem of Brezzi is formulated. Define suitable macros for the norms as well as the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$.

Theorem (Brezzi 1974). Let X and Y be hilbert spaces. Further, let $a : X \times X \to \mathbb{R}$ and $b : X \times Y \to \mathbb{R}$ be continuous bilinear forms and $X_0 := \big\{x \in X \,:\, b(x,\cdot) = 0 \in Y^*\big\}$. Under the assumptions

- $\alpha := \inf\limits_{v \in X_0\setminus\{0\}} \dfrac{a(v,v)}{\|v\|_X^2} > 0$, i.e., $a(\cdot,\cdot)$ is elliptic auf X_0,
- $\beta := \inf\limits_{y\in Y\setminus\{0\}} \sup\limits_{x\in X\setminus\{0\}} \dfrac{b(x,y)}{\|x\|_X\|y\|_Y} > 0$

there holds the assertion: For each $(x^*, y^*) \in X^* \times Y^*$ there is a unique solution $(x, y) \in X \times Y$ of the so-called saddle point problem

$$\begin{array}{lcll} a(x,\widetilde{x}) \;+\; b(\widetilde{x},y) & = & x^*(\widetilde{x}) & \quad\text{for all } \widetilde{x} \in X, \\ b(x,\widetilde{y}) & = & y^*(\widetilde{y}) & \quad\text{for all } \widetilde{y} \in Y. \end{array} \tag{1}$$

Aufgabe 11.4. Write a `myenumerate`-environment with associated counter, which generates for a code

```
\begin{myenumerate}
  \myitem A
  \myitem B
  \myitem C
\end{myenumerate}
```

the following result

(i) A

(ii) B

(iii) C

where the numbering of the roman numbers is automatic. Build on the `itemize`-environment. To this end, write a macro `\myitem`, which uses the command `\item`. Check via the WWW how you could solve this exercise as well with the help of the `enumerate`-package.

Aufgabe 11.5. Write a `theorem`- and a `lemma`-environment with the following layout, cf. Aufgabe 11.1. Here, $\square$ is generated via `\square`. For both environments the same counter should be used. The counter should depend on the chapter and the section. Optionally, one should be able to give the theorem and lemma a name. Use these environments in a document with at least one chapter (`chapter`) and two sections (`section`). Write in each section an arbitrary theorem and an arbirtrary lemma of your analysis lecture (without proofs). Use always an appropriate `\label`.

Satz 1.1.2 (BOLZANO-WEIERSTRASS). In a finite dimensional normed space X, each bounded sequence $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence. $\square$

Lemma 1.1.3 (ZORN). *Suppose a partially ordered set P has the property that every chain has an upper bound in P. Then the set P contains at least one maximal element.* $\square$

Aufgabe 11.6. The matrix $L \in \mathbb{R}^{n\times n}$ has the following form

$$L = \begin{pmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{pmatrix}$$

with $L_{11} \in \mathbb{R}^{k\times k}$ and $0 < k < n$. If L_{11} and L_{22} are regular, then L is regular as well, and the inverse is given by

$$L^{-1} = \begin{pmatrix} L_{11}^{-1} & 0 \\ -L_{22}^{-1}L_{21}L_{11}^{-1} & L_{22}^{-1} \end{pmatrix}.$$

Formulate the result with its proof in LaTeX.

Aufgabe 11.7. For a differentiable function $f : \mathbb{R} \to \mathbb{R}$ and $x \in \mathbb{R}$ we define the **one-sided differential quotient**

$$\Phi(h) := \frac{f(x+h) - f(x)}{h} \quad \text{for } h > 0$$

and $\Phi(0) := f'(x)$. There holds $\lim_{h\to 0} \Phi(h) = \Phi(0)$. Prove with the help of the Taylor theorem that for $f \in C^2(\mathbb{R})$ it holds

$$|\Phi(0) - \Phi(h)| = \mathcal{O}(h),$$

and determine the constant which hides in the Landau notation as accurate as possible. Write your result as a lemma with proof in LaTeX.

Aufgabe 11.8. For $f \in C^2(\mathbb{R})$ it holds $e_h := |\Phi(h) - \Phi(0)| = \mathcal{O}(h)$ with the function Φ of Aufgabe 11.7. For arbitrary $f \in C^1(\mathbb{R})$, one observes only $e_h = \mathcal{O}(h^\alpha)$ for some $\alpha \in (0, 1]$. The constant α is called **convergence order**. With the approach $e_h = ch^\alpha$ the term $\delta_h := |\Phi(h) - \Phi(h/2)|$ satisfies the estimate

$$e_h(1 - 2^{-\alpha}) \le \delta_h \le e_h(1 + 2^{-\alpha}),$$

i.e., it holds as well $\delta_h = \mathcal{O}(h^\alpha)$. With the further approach $\delta_h = Ch^\alpha$ one obtains for h and $h/2$ two equations from which one can calculate the **experimental convergence order** and the corresponding constant C:

$$\alpha = \log(\delta_h/\delta_{h/2})/\log(2) \quad \text{as well as} \quad C = \delta_h/h^\alpha.$$

Formulate the text of the exercise in your own words with all mathematical intermediate steps in LaTeX.