Exercise sheet 1 for the course
"Computational Finance"

Problem 1. Assume $K_1 < K < K_2$.
You buy a call option with strike K_1, a put option with strike K_2 and you sell a call and a put option with strike K.

Illustrate the payoff of the resulting portfolio at expiary and give a short intuition behind such a portfolio.
(Assume all option expire in the exact same moment and ignore the price of the option.)

Problem 2. Let P_t be the value of a European put option at time t.
Show:

$$\left(Ke^{-r(T-t)} - S_t\right)^+ \leq P_t \leq Ke^{-r(T-t)} \quad \text{for } 0 \leq t \leq T.$$

Hint: Proposition 2.3

Problem 3. Let $C(S,t;K)$ be the value of a European call option with strike price K. Show that $K \mapsto C(S,t;K)$ for fixed (S,t) is convex.
Hint: One way is to argue by contradiction and construct an arbitrage opportunity. A good starting point for a contradiction argument would be to play around with different strike prices and look at the resulting portfolio.

Problem 4. Let $\mathbb{G}$ be a given filtration and λ be a $\mathbb{G}$ adapted, non negative, continuous stochastic process. Let $E \sim \text{Exp}(1)$ be an exponentially distributed random variable (with parameter 1) which is independent of $\mathbb{G}$. There exists a suitable filtration $\mathbb{F}$, such that $G_t \subset F_t$ for all t.

We define $\tau := \inf\left\{t \,:\, \int_0^t \lambda_s \, ds \geq E\right\}$. r_t denotes the $\mathbb{G}$ adapted process modelling the interest rate (risk-free rate).
Assume we want to price a zero coupon bond which which pays an amount of 1 at time T, which you can assume to be simply the discounted future value $\mathbb{E}\left[1 * \exp\left(-\int_0^T r_s \, ds\right)\right]$. This however only happens if the bond doesn't default till then. The default time is given by τ.
Calculate the current value of this defaultable bond.

Hint 1: Ignoring any technicalities, the fair price is given by $\mathbb{E}\left[\exp\left(-\int_0^T r_s \, ds\right) 1_{\{\tau > T\}}\right]$. Take the conditional expectation with respect to G_T and use the measurability of certain expressions. The solution will still be of the form $\mathbb{E}[\ldots]$.
Hint 2: $\mathbb{P}(\tau > T | G_T) = \mathbb{P}\left(\int_0^T \lambda_s \, ds < E | G_T\right)$.

Problem 5. You are free to choose any stock (or financial time series) and any time interval (.....obviously more than one data point though). Load the corresponding data into the program of your choice. Plot the price data and calculate the mean and variance. Hint: You can either use the provided data or just go to a site like Yahoo finance and download their "historical data".

Problem 6. Given the interval $[0, 1]$ using N (uniform) time steps (you are free to choose N). We will call them $0 = t_0 < t_1 < \cdots < t_N = 1$

Simulate N independent $N(0, 1)$ distributed random variables Z_i.
Set $B(0) = 0$,
$B(t_1) = \sqrt{t_1} Z_1$, $B(t_2) = B_{t_1} + \sqrt{t_2 - t_1} Z_2$,
until $B(t_N) = \sum_{i=1}^{N} \sqrt{t_i - t_{i-1}} Z_i$.
Plot the vector B.

Hint:
$B = [0, \dots, 0]$
for i from 1 to N
    simulate new $Z \sim N(0, 1)$
    $B[i] = B[i-1] + \sqrt{\frac{1}{N}} * Z$
plot(B)