Exercise sheet 2 for the course "Computational Finance"

Hint. (General) When approaching problems involving the Ito formula, it is often useful to write it out explicitly for an arbitrary function f (here time-independent 1-D),

$$f(W_t) = f(W_0) + \int_0^t f'(W_s)\, dW_s + \frac{1}{2}\int_0^t f''(W_s)\, ds$$

and "match terms".
Recall that the symbolic way of writing this is $df(W_t) = f'(W_t)\, dW_t + \frac{1}{2} f''(W_t)\, dt$.

Problem 1. Let $(W_t)_{t\geq 0}$ be a Wiener process.
1.) Show that $W_t^2 - t$ is a martingale with respect to the natural filtration generated by W_t.
2.) For $n \in \mathbb{N}$, compute $\int_0^t W_s^n\, dW_s$.

Hint. Look at the derivatieves of polynomials. Use the "fact" that the term $\int_0^t \cdot\, dW_s$ is a martingale.
Compare $\int_0^t W_s^n\, dW_s$ to the terms appearing in the Ito formula. "Guess" the right function $f(x)$.

Problem 2. Solve

$$dX_t = -X_t dt + \sigma(t) dW_t, \quad t > 0, \quad X_0 = 0,$$

where W_t is the Wiener process. Determine $\lim_{t\to\infty} \mathrm{E}(X_t)$ and for constant σ, evaluate $\lim_{t\to\infty} \mathrm{Var}(X_t)$.
You can assume without proof that $\mathrm{E}\left[\int_0^t \cdot\, dW_s\right] = 0$.

Hint. 1.) Consider $Y_t := X_t \exp(\theta t)$ and apply Ito's formula (time dependent) to the function $f(t,x) = x \exp(\theta t)$.
This will result in

$$dY_t = f_t(t, X_t)\, dt + f_x(t, X_t)\, (dX_t) + \frac{\sigma(t)^2}{2} f_{xx}\, dt.$$

"dX_t" is given above, so simply plug it in.
2.) Look at the result and choose θ such that the expression gets slightly simpler (you want to cancel one term).
3.) Now consider the integral form of the result and revert the change of variables $X_t = Y_t \exp(-\theta t) = \exp(-\theta t) Y_0 + \dots$.
If you plugged the result into Ito's formula, you would obtain the original SDE (you don't have to do this here)
Use the Ito-isometry for $\mathrm{E}[X_t^2]$.

Problem 3. Consider the interval $[0, T]$ and a given partition, where $|P|$ is the length of the longest subinterval. We define the quadratic covariation (process) of the two processes X and Y by

$$[X, Y]_t := \lim_{|P| \to 0} \sum_{i=1}^{N} (X_{t_i} - X_{t_{i-1}})(Y_{t_i} - Y_{t_{i-1}}).$$

Let f be a Lipschitz continuous (deterministic) function and X be a (stochastic) process with continuous paths (think of the Brownian motion). Then $[f, X]_t = 0$.

Hint. If you fix an ω, the path $X(\omega, t) = X(t)$ is continuous on the closed interval $[0, T]$. Hence it is uniformly continuous (gleichmäßig stetig). You can then use the fact that the total variation of a Lipschitz continuous function is bounded.

Problem 4. Let V be a European call or put option. Compute the *Greeks*

$$\Delta = \frac{\partial V}{\partial S}, \quad \Gamma = \frac{\partial^2 V}{\partial S^2}, \quad \kappa = \frac{\partial V}{\partial \sigma}, \quad \theta = \frac{\partial V}{\partial t}, \quad \rho = \frac{\partial V}{\partial r}.$$

Show that $\theta + \frac{1}{2}\sigma^2 S^2 \Gamma + rS\Delta - rV = 0$. Comment: κ is usually called Vega.

Hint. The Black-Scholes formula gives you an explicit form of V.

Problem 5. Consider the time interval $[0, 1]$ divided into N time steps. Simulate a few sample-paths of the expression:

$$\sum_{i=1}^{N} \sin(t_{i-1})(W_{t_i} - W_{t_{i-1}}).$$

If you don't manage to plot multiple paths, plot them at least individually.
Bonus (not mandatory): Average over the value of the paths at time 1. What do you see/expect to see if you increase the number of simulations?

Hint. $W_{t_{i+1}} - W_{t_i} \sim N(0, \varDelta t) = \sqrt{\varDelta t} N(0, 1)$.

Problem 6. Consider again the time interval $[0, 1]$ divided into N time steps.
Simulate sample paths of the following expression, starting at $X_0 = 0$:

$$X_{t_{i+1}} = X_{t_i} + 5 * (t_i - t_{i-1}) + 3 * (W_{t_i} - W_{t_{i-1}}).$$

Please plot your results.

Hint. If you think about the distribution of the sum of the second and third term on the right hand side, it could make reusing old code easier.