Exercise sheet 3 for the course "Computational Finance"

Problem 1. Let $(S - Ke^{-r(T-t)})^+ \leq C_0 \leq S$ and let

$$C(\sigma) = S\Phi(d_1(\sigma) - Ke^{-r(T-t)}\Phi(d_2(\sigma))$$

be the call option price as a function of the volatility $\sigma \geq 0$. Show that the equation $C(\sigma) = C_0$ has a unique solution.

Hint. For the existence, you can use one of the "Greeks" to show that $C(\sigma)$ is a continuous, strictly increasing function. It might be smart to also consider the limits $\sigma \to \infty$ and $\sigma \to 0$. Regarding the uniqueness, calculate the appropriate "Greeks" ($\frac{\partial^2 C}{\partial \sigma^2} = \frac{d_1 d_2}{\sigma}\frac{\partial C}{\partial \sigma}$) and think about the number of inflection points the function can have.

Problem 2. A digital call option with strike price K and expiry date T has the payoff function $C_T(S) = 1$ if $S > K$ and $C_T(S) = 0$ otherwise. The corresponding digital put option has the payoff $P_T(S) = 0$ if $S > K$ and $P_T(S) = 1$ otherwise. Derive the price of a digital call and put option at time t and show the call-put parity $C(S,t)+P(S,t) = e^{-r(T-t)}$. (Remark: Digital options are also called binary options or cash-or-nothing options.)

Hint. Under the risk neutral measure (which exists due to our assumptions on the market),

$$dS_t = rS_t\,dt + \sigma S_t\,dW_t^Q,$$

where r is the risk free rate and W_t^Q is a Brownian motion under Q.
Calculate $C(t,S) = \exp(-r(T-t))\mathrm{E}^Q[1_{S_T>K}]$. $\frac{\ln S_T - \ln(S_t) - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$ follows which distribution? Can you rewrite the term inside the indicator as something you know.

Problem 3. An exchange option gives the holder the right to exchange one asset for another. The payoff function reads as $V(S_1,S_2,T) = (S_1 - S_2)^+$, and $V(S_1,S_2,t)$ solves the two-dimensional Black-Scholes equation with correlation constant ρ. We assume that the assets pay continuous dividends with rate q_1, q_2, respectively.
(i) Define the function $U(\xi,t)$ by $V(S_1,S_2,t) = S_2U(\xi,t)$ with $\xi = S_1/S_2$. Show that U solves the one-dimensional Black-Scholes equation

$$U_t + \frac{1}{2}\sigma_*^2\xi^2 U_{\xi\xi} + (q_2 - q_1)U_\xi - q_2 U = 0, \quad U(\xi,T) = (\xi - 1)^+,$$

where $\sigma_*^2 = \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2$.
(ii) Assuming the same boundary conditions as for the standard Black-Scholes equations, determine the price of $V(S_1,S_2,t)$.

Hint. Calculate the first and second partial derivatives of $S_2U(\xi,t)$ with respect to S_1, S_2 and t and plug them into the multidimensional Black-Scholes equation.
Use the standard Black-Scholes equation to derive an expression in ξ.

Problem 4. Consider the following parameters: $r = 0.0328$, $T = 0.211$, $S_0 = 5290.36$. Let the strikes K and corresponding call option prices be given by the following table,

K=	6000	6200	6300	6350	6400	6600	6800
Call price =	80,2	47,1	35,9	31,9	27,7	16,6	11,4

Approximate and plot the implied volatility for each option and plot the result.

Hint. Step 1: Implement the Black-Scholes formula as if you wanted to price an option.
Option 1:
Step 2: Choose an initial guess for σ, value the option with the given parameter and calculate the difference to the market price given above.
Step 3:Implement the Newton method in the lecture notes (p. 36) in order to get a new sigma and repeat "Step 2" for 20 iterations or until a certain accuracy is reached .
Option 2:
Step 2: Choose an upper (σ_u) and a lower bound (σ_l) for σ (unrealistically high and low ...2, 0.001).
Step 3: Take the average $\sigma_0 = \frac{\sigma_l + \sigma_u}{2}$ value the option with σ_0. Calculate the difference to the market price given above.
Step 3: Proceed to find the sigma corresponding to the market price by bisection.

Problem 5. Go to "www.barchart.com" (or any other page providing options data). Download the data and and repeat the previous exercise with real world data. You also have to find a good proxy for the risk-free rate.
If your source provides "implied volatility", plot your own results against the given ones.

Hint. Be careful regarding the expiration date. You might have to re scale your risk-free rate.

Remark 1. There was a question regarding the distribution of $\int_0^t W_s^n \, ds$ during the last exercise and I attempt to provide a more satisfying answer:
What we can do/see easily with the tools from the lecture and measure theory 1+2:
1.) Existence of moments and bounds for them.
2.) If we have a Gaussian process Y_t, then $\int_0^t Y_s \, ds$ is Gaussian as well.
3.) By using Feynman-Kac, technically, we could express the quantity $\mathrm{E}\left[\exp\left(-\int_0^t f(W_s)\, ds\right)\right]$.
What can we calculate explicitly:
1.) The expectation (Fubini)
2.)σ: The easiest way I came up with involves quite a bit of combinatorics. For those with too much spare time: Write the square as a double integral and use Lemma 4.5 of the paper "Sample path properties of the local times of strongly symmetric Markov processes via Gaussian processes" by "Michael Marcus" and "Jay Rosen".

Describing the distribution in a satisfactory way seems to be not the easiest of tasks.