Exercise sheet 4 for the course
“Computational Finance”
(first) Corrected version

Problem 1. Consider the stock price dynamics given by

$$dS_t = (r-q)S_t\,dt + \sigma_l(t,S_t)S_t\,dW_t,$$

where σ_l is now a function of the time and stock price. Let $C = C(K, T-t)$ be the price of a call option, given as a function of strike and time to maturity. For simplicity, let's assume $t = 0$.
Show that the local volatility function σ_l then satisfies

$$\sigma_l^2(T,K) = \frac{\frac{\partial C}{\partial T} + (r-q)K\frac{\partial C}{\partial K} + qC}{\frac{K^2}{2}\frac{\partial^2 C}{\partial K^2}}.$$

You can assume that the probability density function $f(x,t)$ of the stock price at time t evaluated at $S_t = x$ satisfies the following equation (Kolmogorov forward equation):

$$-\frac{\partial f}{\partial t} - (r-q)\frac{\partial xf}{\partial x} + \frac{1}{2}\frac{\partial^2 \sigma(t,x)^2x^2 f}{\partial x^2} = 0,$$

with $p(x,t) = \delta_{S_0}(x)$ at $t = 0$.

Hint. $C(K,T) = \exp(-rT)\mathrm{E}^{\mathbb{Q}}[(S_T-K)^+] = \exp(-rT)\int_K^\infty (x-K)f(x,T)\,dx$.
Differentiate this the expression twice with respect to K (obtaining C_{KK}, then with respect to T (C_{KKT}). Use the given equation above and substitute and substitute parts of the appearing terms ($C_{KK} = \dots$). Integrate with respect to K twice and argue why the integration constants $c_1(T)$, $c_2(T)$ will vanish.

Problem 2. Consider the following Black-Scholes equation for an Asian option:

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + SV_I + rSV_S - rV = 0, \quad S>0,\ I>0,\ t\in(0,T),$$

with a linear payoff, $V(S,I,T) = a + bS + cI$. Find a special solution by using the ansatz $V(S,I,t) = \alpha(t) + \beta(t)S + \gamma(t)I$ and determine $\alpha(t)$, $\beta(t)$, and $\gamma(t)$.

Hint. Plug in and solve the ODEs.

Problem 3. Let $(B_t^H)_{t\geq 0}$ be a fractional Brwonian motion with Hurst parameter $H \in (0,1)$. Let $X_n := B_n^H - B_{n-1}^H$ for $n\in\mathbb{N}$. Show that
(i) $\mathrm{Cov}(X_{n+1}^H, X_1^H) = \frac{1}{2}((n+1)^{2H} + (n-1)^{2H} - 2n^{2H})$.
(ii) $\mathrm{Cov}(X_{n+1}^H, X_1^H)$ behaves like $H(2H-1)n^{2H-2}$ as $n\to\infty$, i.e.

$$\lim_{n\to\infty}\frac{\mathrm{cov}(X_{n+1}^H, X_1^H)}{H(2H-1)n^{2H-2}} = 1.$$

(iii) Looking at $\mathrm{E}[(B_{t_1}^H - B_{s_1}^H)(B_{t_2}^H - B_{s_2}^H)]$ for $t_2 > s_2 > t_1 > s_1 > 0$, what happens for $H\in(0,\frac{1}{2})$ and $H\in(\frac{1}{2},1)$ and how would you interpret this behaviour.

Hint. Consider the function $f(x) = \left((1-x)^{2H} + (1+x)^{2H} - 2\right)$ (*cough* Taylor). Rescale it to $\frac{1}{2}n^{2H} f\left(\frac{1}{n}\right)$.

Problem 4. Under the Heston model, the price of a call option can be written as

$$C(t,S,v) = Sf_1(t,S,v) - K\exp\left(-r(T-t)\right)f_2(t,S,v),$$

$$f_j = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left(\frac{\psi_j(w,S,v,t)\exp\left(-iw\log(K)\right)}{iw}\right)\,dw,$$

$$\psi_j = \exp(C_j((T-t),w) + vD_j(T-t,w) + iw\log(S)).$$

The functions $C_1,\ C_2$ and $D_1,\ D_2$ are connected by the following equality:

$$C_1(T-t,w) + vD_1(T-t,w) + iw\log(S) + r(T-t) + \log(S)$$
$$= C_2(T-t,w-i) + vD_2(T-t,w-i) + i(w-i)\log(S),$$

hence,

$$\psi_2(w-i,S,v,t) = S\exp(r(T-t))\psi_1(w,S,v,t).$$

Compute "delta", $\frac{\partial C}{\partial S}$.
(!!!! Just calculate up to the substitution. I expected the last step to be easier. !!!!)

Hint. For $\Phi : \mathbb{R} \to \mathbb{C}$, $Re\left(\frac{d}{dx}\Phi(x)\right) = \frac{d}{dx}Re\left(\Phi(x)\right)$.
Express $\frac{\partial f_1}{\partial S}$ in terms of ψ_2.

Problem 5. 1.) Simulate a Poisson process N with intensity $\lambda = 1$. This is a process with $N_t - Ns \sim Poi(\lambda(t-s))$.
2.) Implement a compound Poisson process with normal jump distribution (this corresponds to ϕ on page 52 of your lecture notes.

Hint. You can either sample a Poisson process the obvious way, or by using that the jumps themselves arrive at increments of time Δt according to the exponential distribution $E(\lambda)$. Hence N_t can be simulated by sampling $E(\lambda)$ variables, until the time increments sum to t.