Exercise sheet 5 for the course
"Computational Finance"

Problem 1. We can derive an alternative binomial method by solving

$$e^{r\triangle t} = pu + (1-p)d, \quad e^{(2r+\sigma^2)\triangle t} = pu^2 + (1-p)d^2, \quad p = \frac{1}{2}.$$

Show that

$$d = e^{r\triangle t}\big(1 - \sqrt{e^{\sigma^2\triangle t} - 1}\big), \quad u = e^{r\triangle t}\big(1 + \sqrt{e^{\sigma^2\triangle t} - 1}\big).$$

Determine bounds on $\triangle t$ which ensure that $d > 0$ and $u > 0$.

Problem 2. Let $s = \frac{1}{2}(e^{-r\triangle t} + e^{(r+\sigma^2)\triangle t})$. Show that

$$u = s + \sqrt{s^2 - 1} = e^{\sigma\sqrt{\triangle t}} + O(|\triangle t|^{3/2}) \quad \text{as } \triangle t \to 0.$$

Problem 3. Consider the equation (Cox-Ingersoll-Ross),

$$dX_t = \kappa(\theta - X_t)\,dt + \sigma\sqrt{X_t}\,dW_t.$$

Assume $2\kappa\theta \geq \sigma^2$ and $X_0 > 0$.
We know that the anaytic solution remains positive with probability 1 (since it follows a chi-squared distribution).
1.) Show that the probability that the Euler discretization produces negative values is larger than 0,
2.) Show that for any SDE of the form

$$X_{n+1} = X_n + a(t_n, X_n)\Delta t + b(t_n, X_n)\Delta W,$$

the Euler discretization doesn't satisfy $\mathbb{P}(X_{n+1} > 0 | X_n > 0) = 1$.

Problem 4. Let the usual Hermite polynomials be given by

$$H_n(x) := \frac{(-1)^n}{n!}\exp\left(\frac{x^2}{2}\right)\frac{d^n}{dx^n}\exp\left(-\frac{x^2}{2}\right).$$

Let us consider the time dependent version (which is equal to $t^{\frac{n}{2}}H\left(\frac{x}{\sqrt{t}}\right)$),

$$H_n(t,x) := \frac{(-t)^n}{n!}\exp\left(\frac{x^2}{2t}\right)\frac{d^n}{dx^n}\exp\left(-\frac{x^2}{2t}\right).$$

It is known (by messy calculations, which don't need to be done at this point) that,

$$\begin{aligned}
H_{n+1} &= \frac{x}{n+1} H_n(t,x) + \frac{t}{n+1} H_{n-1}(t,x), \\
\frac{d}{dt} H_n(t,x) &= -\frac{1}{2} H_{n-2}(t,x), \\
\frac{d}{dx} H_n(t,x) &= H_{n-1}(t,x).
\end{aligned}$$

Show that

$$dH_{n+1}(t,W) = H_n(t,W).$$

Use this easy result to show

$$H_n(t,W) = \int_0^t \int_0^{s_n} \cdots \int_0^{s_2} dW_{s_1} \ldots dW_{s_{n-1}} \, dW_{s_n}.$$

Further assume $f \in L^2(0,T)$ and $t \leq T$. Show that

$$\int_a^t \int_a^{s_n} \cdots \int_a^{s_2} f(s_1) \ldots f(s_n) \, dW_{s_1} \ldots dW_{s_n} = H_n \left(\int_a^t f^2(s) \, ds, \int_a^t f(s) \, dW_s \right).$$

Problem 5. Implement an Euler discretization for the following SDE:

$$dS_t = rS_t \, dt + \sigma S_t \, dW_t,$$

with $S_0 = 10, K = 12, r = 0.04, \sigma = 0.4, T = 2$. Calculate $\mathrm{E}\left[\exp(-rT)(K - S_T)^+ 1_{(S_t < H \; \forall t \in [0,T])}\right]$ for $H = 13$. This corresponds to a European up&out option, which becomes worthless if the price process crosses the barrier H.
The way to approximate this value is to simulate a large amount of paths, setting the payoff of the ones crossing over H to 0, and averaging the payoffs at the exercise time T. The analytic solution yields the price $P = 2.047849$.

Problem 6. Implement the Binomial model and value the European call with the following parameters: $S = 50, r = 0.1, K = 50, \sigma = 0.4, T = 0.411$.
Compare the result to the one obtained by the explicit Black-Scholes formula.