Exercise sheet 6 for the course "Computational Finance"

Problem 1. Let W_t be the Wiener process and let $\triangle Z = \int_{t_i}^{t_{i+1}} \int_{t_i}^{s} dW_\tau ds$. Show that

$$\mathrm{E}(\triangle Z) = 0, \quad \mathrm{E}(\triangle Z)^2 = \frac{h^3}{3}, \quad \mathrm{E}(\triangle W \triangle Z) = \frac{h^2}{2}.$$

Problem 2. Let Y_1, Y_2 be two independent $N(0,1)$-distributed random variables. Show that

$$\triangle W := \sqrt{h} Y_1, \quad \triangle Z := \frac{1}{2} h^{3/2} \left(Y_1 + \frac{Y_2}{\sqrt{3}} \right)$$

satisfy $\mathrm{E}(\triangle Z) = 0$, $\mathrm{E}(\triangle Z)^2 = h^3/3$, and $\mathrm{E}(\triangle W \triangle Z) = h^2/2$.

Problem 3. Let $\lambda > 0$ and $\sigma > 0$ and $Y_0 = 0$ and set

$$Y_{i+1} = Y_i - \frac{\lambda}{2}(Y_i + Y_{i+1})\triangle t + \sigma \triangle W,$$

where $\triangle W = \sqrt{h} Z$ and $h > 0$, $Z \sim N(0,1)$. Show that $\mathrm{E}(Y_i^2) \to \sigma^2/(2\lambda)$ as $i \to \infty$.

Problem 4. Let P denote the payoff (or other output functional of interest) of some financial instrument. Given $l = 1, \ldots, L$ the "levels of resolution" (a sequence of partitions of a time interval $[0, T]$), with $l = 0$ being the coarsest. With P_l we denote the approximation at level l. You can assume that the finest level L has 2^L uniform time steps ($\Delta t_L = 2^{-L} T$).

$$\mathrm{E}[P_L] = \mathrm{E}[P_0] + \sum_{i=1}^{L} \mathrm{E}[P_l - P_{l-1}].$$

Let $V_l = \mathrm{Var}(\mu_l)$.
Given a computational cost C, find the numbers N_l of simulations at level l which minimizes the variance $\mathrm{Var}(\bar{\mu}) = \sum_{l=0}^{L} \frac{V_l}{N_l}$ under the condition $\mathrm{cost}(\bar{\mu}) = \sum_{l=0}^{L} \frac{N_l}{\Delta_l t} = C$.
How does the resulting expression look, if you want $\mathrm{Var}(\bar{\mu}) \leq \frac{\epsilon^2}{2}$?

Hint. Treat N_l as a real valued variable. Analysis 2.

Problem 5. Price the following call option under the Heston model (p.44 of the lecture notes): $S = 100, K = 100, r(= \mu) = 0.05, \lambda = 0.3, \theta = 0.04, \rho = -0.5, \kappa = 1.2, T = 1$. How would you fix problems which might appear?