Exercise sheet 7 for the course "Computational Finance"

Problem 1. Let $u : \mathbb{R} \to \mathbb{R}$ be a smooth function and $h \in \mathbb{R}$. Show that

$$u_{xx}(x) = \frac{1}{h^2}\Big(- \frac{1}{12}u(x+2h) + \frac{4}{3}u(x+h) - \frac{5}{2}u(x) + \frac{4}{3}u(x-h) - \frac{1}{12}u(x-2h)\Big) + O(h^4).$$

Problem 2. Let $u \in C^2(\mathbb{R})$ be a solution to $u_{xx} = f$, where $f \in C^0(\mathbb{R})$, and let w_i be an approximation of $u(x_i)$, where $x_i = ih$ and $i \in \mathbb{Z}$, $h > 0$. By evaluating $u_{xx} = f$ at x_i and $x_{i\pm1}$, determine the consistency error of the compact finite-difference scheme

$$\frac{1}{h^2}(w_{i+1} - 2w_i + w_{i-1}) = \frac{1}{12}\big(f(x_{x+1}) + 10f(x_i) + f(x_{i-1})\big).$$

Hint. If you are stuck, you are allowed to relax the assumptions on f to C^2.

Problem 3. Let $G = (-R, R)$, for an $R > 0$, be an open set and let $\tau_G := \inf\{t \geq 0 \mid X_t \in G^c\}$. This corresponds to the first time when the process hits G^c. Then the price of a knock-out barrier option with payoff $g(\exp(x))$ is given by

$$V_R(t,x) = \mathrm{E}\left[\exp(-r(T-t))g(\exp(X_T))1_{\{T<\tau_G\}}|X_t = x\right].$$

For an option without the barrier is assumed to be given by

$$V(t,x) = \mathrm{E}\left[\exp(-r(T-t))g(\exp(X_T))|X_t = x\right].$$

Show that there exist constants $C(t,\sigma)$, c_1, $c_2 > 0$ such that

$$|V(t,x) - V_R(t,x)| \leq C(T,\sigma)\exp(-c_1 R + c_2|x|).$$

We assume that there exist a $C > 0$, $q \geq 1$ such that $g(s) \leq (s+1)^q$ for all $s \in \mathbb{R}$.
Further let $dX_t = d\log(S_t) = \left(r - \frac{1}{2}\sigma^2\right)dt + \sigma dW_t$.

Hint. If you are trying to show

$$E\left[g\left(\sup_{s\in[0,T]}|X_s|\right)\right] < \infty,$$

it suffices to show that

$$E\left[g(|X_T|)\right] < \infty,$$

if $g(x)$ is a nonnegative, continuous, submultiplicative function on $[0,\infty)$ which goes to ∞ for $x \to \infty$. You don't have to check that this is satisfied by our g.
Use the fact that the expectation is simply an integral with respect to a density $f_{T-t}(\cdot)$.

Remark 1. For the next problems, you can choose θ freely. (Explicit methods are easier to implement). Start with small space and time increments to test/understand the code you are using.

Problem 4. Consider the boundary value problem:

$$-\frac{\partial^2 u}{\partial x^2} = f \text{ in } \Omega = (0,1),$$
$$u(0) = u(1) = 0.$$

Take an equidistant mesh on $(0,1)$.
Discretize the equation and simulate the solution for $f(x) = sin(\pi x)$.

Problem 5. Consider the boundary value problem:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \text{ in } \Omega = (0,1) \times (0,1)$$
$$u(x,0) = \sin(\pi x),$$
$$u(0,t) = u(1,t) = 0.$$

Consider an equidistant mesh on $(0,1)$.
Discretize the equation and simulate the solution at point $T = 1$.

Hint. Explicit methods are usually more intuitive to implement, since the solution process involves just 2 loops: 1 for the time and 1 for the space variable.
There is a stability condition attached to the problem, you choose $\Delta t \leq (\Delta x)^2$ to avoid it. (Courant-Friedrichs-Levy condition).