Exercise sheet 8 for the course “Computational Finance”

Remark 1. During the whole exercise b^* denotes the optimal exercise barrier.
When considering stopping times, it suffices to think about (first) hitting times (of a certain level) for this exercise.

Problem 1. Consider an America future contract, which is a contract with payoff $S_T - K$ at maturity T (like an option, but NOT the max-value). This contract can be exercised at any time $t < T$. Show that it is not optimal to exercise this contract early.
How would you argue if the payoff was $(S_t - K)_+$?
Bonus: How would you argue in the case $(K - S_t)_+$?

Hint. You are in the ”Black-Scholes world”.
The value is obviously given by $V = \sup_{0\leq\tau\leq T} \mathrm{E}^Q[\exp(-r\tau)(S_\tau - K)]$, where τ is a stopping time. What does the Optional Stopping Theorem tell you? (and at which point in time is it usually most comfortable to calculate an expectation)

Problem 2. Let $\tau := \inf\{t : S_t \geq b\}$, where S_t is defined as in the previous problem (this time whitout the expiration date). Assume $\mathbb{P}(\tau < \infty) = 1$ (can be shows via reflection arguments combined with brute force calculations or Girsanov's theorem). Calculate

$$\mathrm{E}\left[\exp(-r\tau)\right]$$

Hint. $\lim_{t\to\infty} \mathrm{E}[\exp(t \wedge \tau) S_{t\wedge\tau}] = \dots$ when everything is nice and you use the assumption imposed on τ.

Problem 3. Consider a perpetual American option (that is an option without maturity date, hence independence of time) and the following free-boundary problem:

$$\begin{aligned}
&\frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{S^2} + (r-\delta) S \frac{\partial V}{S} - rV(S) = 0,\ 0 < S < b^* \\
&V(b^*) = 1 \\
&V'(b^*) = \beta \\
&V(0) = 0 \\
&V(S) < 1\ S < b^* \\
&V(S) = 1\ S > b^* \\
&\beta > 0
\end{aligned}$$

Find the value of the boundary b^*,
Solve the free-boundary problem and find $V(S)$

What happens when $\beta \to 0$?
State the financial significance of this problem (what kind of option are you looking at).

Hint. Look for solutions as powers of S. Consider a linear combination of these solutions and determine the coefficients from the boundary conditions. You should get an equation for b^*.

Problem 4. With the notation from the lecture notes (section 6.4), show that the discrete linear complementarity problem

$$(Aw - b) \cdot (w - f) = 0, \quad Aw - b \geq 0, \quad w - f \geq 0$$

is equivalent to the minimization problem: Find $w \in M$ such that

$$J(w) = \min_{v \in M} J(v), \quad \text{where} \quad J(v) = \frac{1}{2} v^\top A v - b^\top v$$

and $M = \{v \in \mathbb{R}^{n-1} : v_i \geq f_i \text{ for all } i\}$.

Hint. Sometimes it is good to vary a function by $\pm\epsilon$.

Problem 5. Use the finite difference method (code from the last exercise) to price the following European call:
$S_0 = 10,\ K = 12,\ r = 0.04,\ \sigma = 0.4,\ T = 2.$

Hint. 1.) You can either alter the code you already have to fit the problem (as outlined on page 104)
OR
2.) You can transform the pricing problem to the heat equation (see: Proof of the Black-Scholes formula, page 28)