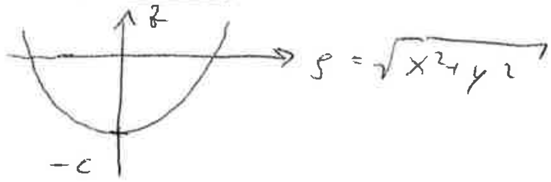
Analysis II, 2. Übung, SS 2012 BS

1) a) $z = x^2+y^2-c = \rho^2 - c$



b) $\vec v_1 = \begin{pmatrix}1\\0\\ \partial z/\partial x\end{pmatrix} = \begin{pmatrix}1\\0\\2x\end{pmatrix} = \begin{pmatrix}1\\0\\2\end{pmatrix}$

$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\1\\2\end{pmatrix}$ $\quad \vec v_2 = \begin{pmatrix}0\\1\\ \partial z/\partial y\end{pmatrix} = \begin{pmatrix}0\\1\\2\end{pmatrix}$

$c = 1^2+1^2-2 = 0$

$\vec\nabla f = \begin{pmatrix}2x\\2y\\-1\end{pmatrix} = \begin{pmatrix}2\\2\\-1\end{pmatrix} \to \vec\nabla f\cdot\vec v_1 = \vec\nabla f\cdot\vec v_2 = 0$

c) $\vec v = \frac{1}{\sqrt{4+4+1}}\begin{pmatrix}2\\2\\-1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}2\\2\\-1\end{pmatrix}$

$D_{\vec v} f = \vec v\cdot\vec\nabla f = 3$

2) $\partial T/\partial t = \vec\nabla T\,\partial\vec c/\partial t = \frac{1}{\sqrt{x^2+y^2+z^2}}\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}-y\\x\\1\end{pmatrix} = \frac{z}{\sqrt{x^2+y^2+z^2}}\Big|_{\substack{z=t\\x=\cos t\\y=\sin t}} = \frac{t}{\sqrt{1+t^2}}$

$= \frac{\partial}{\partial t}\left(\sqrt{1+t^2}\right) = \frac{t}{\sqrt{1+t^2}}$ Anm: $\partial\vec c/\partial t = \begin{pmatrix}-\sin t\\ \cos t\\ 1\end{pmatrix} = \begin{pmatrix}-y\\x\\1\end{pmatrix}$ ✓

3) $\vec r = \begin{pmatrix}x\\y\end{pmatrix} = r\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}$, $\partial_r\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}$, $\frac{1}{r}\partial_\varphi\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-\sin\varphi\\ \cos\varphi\end{pmatrix}$

a) $g(r,\varphi) = f(x,y) = f(\vec r)$

$\partial_r g = \vec\nabla f\cdot\frac{\partial\vec r}{\partial r}$, $\frac{1}{r}\partial_\varphi g = \vec\nabla f\,\frac{1}{r}\frac{\partial\vec r}{\partial\varphi} \to \begin{pmatrix}\partial_r\\ \frac{1}{r}\partial_\varphi\end{pmatrix} g = \begin{pmatrix}\cos\varphi & \sin\varphi\\ -\sin\varphi & \cos\varphi\end{pmatrix}\vec\nabla f \to$

$\to \vec\nabla f = \begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}\begin{pmatrix}\partial_r g\\ \frac{1}{r}\partial_\varphi g\end{pmatrix}$

b) $\vec\nabla f = 2\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}\begin{pmatrix}\partial_r\\ \frac{1}{r}\partial_\varphi\end{pmatrix} r^2 = \begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}\begin{pmatrix}2r\\0\end{pmatrix} =$

$= 2\begin{pmatrix}r\cos\varphi\\ r\sin\varphi\end{pmatrix}$ ✓

4) $\vec\nabla_{\xi,\eta} g = \underbrace{\vec\nabla_{\xi,\eta}\begin{pmatrix}x\\y\end{pmatrix}}_{=\begin{pmatrix}\partial x/\partial\xi & \partial y/\partial\xi\\ \partial x/\partial\eta & \partial y/\partial\eta\end{pmatrix}}\vec\nabla_{x,y} f = \begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}\vec\nabla_{x,y} f$

$\underbrace{\left(\vec\nabla_{\xi,\eta} g\right)^T\left(\vec\nabla_{\xi,\eta} g\right)}_{=\left(\partial_\xi^2 g+\partial_\eta^2 g\right)} = \left(\vec\nabla_{x,y} f\right)^T\underbrace{\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}}_{=1}\left(\vec\nabla_{x,y} f\right) =$

$= \left(\vec\nabla_{x,y} f\right)^T\left(\vec\nabla_{x,y} f\right) = \left(\partial_x^2 f + \partial_y^2 f\right)$

5) a) $\vec{\nabla}\rho = \phi$

b) $\rho(\vec{x}) = \frac{1}{2} a_{ij} x_i x_j \quad , a_{ij} = a_{ji}$

$\partial_k \rho(\vec{x}) = a_{kj} x_j \Big|_{x_j=\phi} = \underline{\underline{\phi}}$ → nicht eindeutig wenn $\det(a) = \phi$

(z.B.: $a = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \rightarrow \rho = \frac{1}{2} x^2 + xy + \frac{1}{2} y^2 \rightarrow \vec{\nabla}\rho = \begin{pmatrix} x+y \\ x+y \end{pmatrix}$

$\vec{\nabla}\rho = \vec{\phi} \quad \forall\, x = -y$)

c) $H_{ij} = \partial_i \partial_j \rho = \underline{\underline{a_{ij}}}$

$\rho(\vec{\phi}) = \underline{\underline{\phi}}$

$\underbrace{\rho(\vec{\phi})}_{=\phi} + \vec{x}\, \underbrace{\vec{\nabla}\rho(\vec{\phi})}_{=\phi} + \frac{1}{2} \vec{x}^T \underbrace{H(\vec{\phi})}_{=a} \vec{x} = \underline{\underline{\frac{1}{2} \vec{x}^T a \vec{x} = \rho(\vec{x})}}$ → es fehlt kein Restglied

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6) → $f(\vec{x}) = x_1 + x_1 x_2 + x_1 x_2 x_3$ → $H = \begin{pmatrix} \phi & 1+x_3 & x_2 \\ 1+x_3 & \phi & x_1 \\ x_2 & x_1 & \phi \end{pmatrix}$

→ $\vec{\nabla} f(\vec{x}) = \begin{pmatrix} 1 + x_2 + x_2 x_3 \\ x_1 + x_1 x_3 \\ x_1 x_2 \end{pmatrix}$ → z.B.: $\vec{x}_0 = \vec{\phi}$: $f(\vec{x}) \simeq x_1 + x_1 x_2$

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7) $\vec{r}_0 = \begin{pmatrix} 1/e \\ -1 \end{pmatrix}$, $f(\vec{r}_0) = 1 + 1 = \underline{\underline{2}}$, $\vec{\nabla} f = \begin{pmatrix} y/x + e^{y+2} \\ \ln(x) + x e^{y+2} \end{pmatrix} = \underline{\underline{\vec{\phi}}}$

$H = \begin{pmatrix} -y/x^2 & 1/x + e^{y+2} \\ 1/x + e^{y+2} & x e^{y+2} \end{pmatrix} = \begin{pmatrix} e^2 & 2e \\ 2e & 1 \end{pmatrix}$

$f(x_0 + h_x, y_0 + h_y) = \underline{\underline{2 + e^2/2\, h_x^2 + 2e\, h_x h_y + 1/2\, h_y^2}}$

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8) $\vec{\nabla} f = \begin{pmatrix} 2x + y - 1 \\ x - 1 + 3y^2 \end{pmatrix} \stackrel{!}{=} \vec{\phi}$ → $\underline{\underline{\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1/4 \\ 1/2 \end{pmatrix}}}$ & $\underline{\underline{\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2/3 \\ -1/3 \end{pmatrix}}}$

$\partial_x^2 f = 2 > \phi$, $\partial_y^2 f = 6y$, $\partial_x \partial_y f = 1$

↳ $f_{xx} f_{yy} - f_{xy}^2 = 12y - 1 = 5 > \phi$ → Minimum

↳ $f_{xx} f_{yy} - f_{xy}^2 = \underline{-5 < \phi}$ → Sattelpunkt

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9) $\int_0^1 \left[(at+b)^2 - 2(at+b) e^t + e^{2t} \right] dt =$

$= \frac{1}{a} \frac{(at+b)^3}{3} \Big|_\phi^1 \underbrace{- 2a \int_\phi^1 t e^t}_{= -2a(t-1)e^t |_\phi^1} - 2b e^t \Big|_\phi^1 + \frac{1}{2} e^{2t} \Big|_\phi^1 =$

$= \underbrace{\frac{(a+b)^3}{3a} - \frac{b^3}{3a}}_{= \frac{a^2}{3} + ab + b^2} - 2a - 2b(e-1) + \frac{1}{2}(e^2 - 1) = \varphi(a, b)$

$\partial_a \varphi = \frac{2}{3} a + b - 2 \stackrel{!}{=} \phi$ & $\partial_b \varphi = a + 2b - 2(e-1) \stackrel{!}{=} \phi$ → $\underline{a = 18 - 6e}$ & $\underline{b = 4e - 10}$

$\underbrace{\partial_a^2 \varphi = \frac{2}{3} > \phi, \ \partial_a \partial_b \varphi = 1, \ \partial_b^2 \varphi = 2 > \phi}$ $\quad \partial_a^2 \varphi\, \partial_b^2 \varphi - (\partial_a \partial_b \varphi)^2 = \frac{1}{3} > \phi$ → $\underline{\underline{\text{Minimum}}}$

↳ Hesse-Matrix ist konstant

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10) a) $\partial_1 f = \begin{pmatrix} 1 \\ e^{x_1 + x_2} \end{pmatrix}$, $\partial_2 f = \begin{pmatrix} 2x_2 \\ e^{x_1 + x_2} \end{pmatrix}$ → $Df = \begin{pmatrix} 1 & 2x_2 \\ e^{x_1 + x_2} & e^{x_1 + x_2} \end{pmatrix} \Big|_{x_1 = \phi, x_2 = \phi} = \begin{pmatrix} 1 & \phi \\ 1 & 1 \end{pmatrix}$ & $\underline{\underline{\det\begin{pmatrix} 1 & \phi \\ 1 & 1 \end{pmatrix} = 1;}}$

b) $D(f^{-1}) = (Df)^{-1} = \underline{\underline{\begin{pmatrix} 1 & \phi \\ -1 & 1 \end{pmatrix}}}$