1a) $\partial_i(fg) = (\partial_i f)g + f(\partial_i g)$, $\partial_i\left(\frac{f}{g}\right) = \frac{g(\partial_i f) - f(\partial_i g)}{g^2}$

b) $\partial_t\left(\sum_i f_i g_i\right) = \sum_i (\partial_t f_i) g_i + \sum_i f_i(\partial_t g_i)$

c) $\partial_t \sqrt{x_i x_i} = \frac{1}{2}\frac{1}{\sqrt{x_i x_i}}(x_i' x_i + x_i x_i') = \frac{\vec x' \cdot \vec x}{\|\vec x \cdot \vec x\|} = \partial_t \|\vec x\|$

2a) $\vec x' = A\vec x \to \partial_t\|\vec x\| = \frac{\vec x\cdot A\vec x}{\vec x\,\vec x}$

b) $A = -A^T \to \vec x\cdot A\vec x = \vec x\cdot(-A^T)\vec x = -\vec x\cdot A^T\vec x = -\vec x\cdot A\vec x = 0 \to \partial_t\|\vec x\| = 0$

c) $\ddot u + \omega^2 u = 0$, $v = \frac{\dot u}{\omega} \to \dot u = \omega v$

$\omega\dot v + \omega^2 u = 0 \to \dot v = -\omega u$ $\Big\} \to \begin{pmatrix}\dot u\\ \dot v\end{pmatrix} = \begin{pmatrix}0 & \omega\\ -\omega & 0\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}$

3) a) $\|ABx\| \le \|A\|\|Bx\| \le \|A\|\|B\|\|x\|$

$\|ABx\| \le \|AB\|\|x\| \le \|A\|\|B\|\|x\| \to \|AB\| \le \|A\|\|B\|$

b) $\|Ax\|_\infty = \left|\sum_j a_{i_{max}j} x_j\right| \le \sum_j |a_{i_{max}j}||x_j| \le \sum_j |a_{i_{max}j}||x_{k_{max}}| = \sum_j |a_{i_{max}j}|\,\|\vec x\|_\infty \quad \forall \vec x$

für $\vec x = (\pm 1, \dots) = (\mathrm{sign}(a_{i_{max}1}), \dots)$: $\|Ax\|_\infty = \left|\sum_j \underbrace{a_{i_{max}j}\, x_j}_{=|a_{i_{max}j}|}\right| = \sum_j |a_{i_{max}j}| = \sum_j |a_{i_{max}j}|\underbrace{\|\vec x\|_\infty}_{=1} \to$

$\to \|A\|_\infty = \sum_j |a_{i_{max}j}|$

c) $\rho \le \|A^T A\|_1 \le \underbrace{\|A^T\|_1}_{=\sum_i |a_{ji}| = \|A\|_\infty}\|A\|_1 = \|A\|_\infty\|A\|_1 \to \|A\|_2 = \sqrt{\rho} = \sqrt{\|A\|_\infty\|A\|_1}$

d) $A = \begin{pmatrix}-2 & 1\\ 0 & -3\\ 1 & 2\end{pmatrix}$ $\quad A^T A = \begin{pmatrix}-2 & 0 & 1\\ 1 & -3 & 2\end{pmatrix}\begin{pmatrix}-2 & 1\\ 0 & -3\\ 1 & 2\end{pmatrix} = \begin{pmatrix}5 & 0\\ 0 & 14\end{pmatrix}$

$\|A\|_\infty = 3$

$\|A\|_1 = 6$ $\quad \sqrt{14} < \sqrt{18}$ ✓

4) $Df\big|_{\vec r = 0} = A - I + \left.\begin{pmatrix} e^{yz} & xz\,e^{yz} & xy\,e^{yz}\\ zy\,e^{zx} & e^{zx} & xy\,e^{zx}\\ zy\,e^{xy} & xz\,e^{xy} & e^{xy}\end{pmatrix}\right|_{\vec r = 0} = \underline{A}$ $\quad D(f^{-1}) = (Df)^{-1} = A^{-1}$

5) $f = x^3 + x^2y^2 - 14x - 5y$

$\vec\nabla f = \begin{pmatrix}3x^2 + 2xy^2 - 14\\ 2x^2y - 5\end{pmatrix} \overset{!}{=} \begin{pmatrix}0\\ 0\end{pmatrix}$ $\quad D(\vec\nabla f) = \begin{pmatrix}6x + 2y & 4xy\\ 4xy & 2x^2\end{pmatrix}$

$D(\vec\nabla f)\big|_{(x_n,y_n)}\begin{pmatrix}\Delta x_n\\ \Delta y_n\end{pmatrix} = \vec\nabla f\big|_{(x_n,y_n)} \to \begin{pmatrix}x_{n+1}\\ y_{n+1}\end{pmatrix} = \begin{pmatrix}x_n\\ y_n\end{pmatrix} + \begin{pmatrix}\Delta x_n\\ \Delta y_n\end{pmatrix}$

$(x_0, y_0) = (0.80, 0.21) \to (x_1, y_1) = (2.90, 2.80) \to (x_2, y_2) = (2.07, 1.90)$

6) a) i) $\varphi(x_0, y_0) = 0$ ii) φ_x & φ_y stetig in Umgebung von (x_0, y_0) iii) $\varphi_y \neq 0$

b) $\varphi = (1-4x)y^4 - 1$ $\quad$ i) $\varphi(x_0,y_0) = 0$ ✓ ii) $\varphi_x = -4y^4$ & $\varphi_y = 4(1-4x)y^3 \to$ stetig ✓ iii) $\varphi_y \neq 0 \to x \neq \frac{1}{4}$ & $y \neq 0$

$(x_0, y_0) = (0, 1)$

c) $d_x\varphi = \partial_x\varphi + \partial_y\varphi\, y' = 0 \to y' = \partial_y\varphi/\partial_x\varphi = \left.\frac{-4(1-4x)y^3}{-4y^4}\right|_{x_0,y_0} = 1 = y'$

$d_x^2\varphi = \partial_x^2\varphi + \partial_x\partial_y\varphi\, y' + (\partial_x\partial_y\varphi + \partial_y^2\varphi\, y')y' + \partial_y\varphi\, y'' = 0 \to y'' = 5$

$y = y_0 + y'x + y''\frac{x^2}{2} = 1 + x + \frac{5}{2}x^2$

8)
$f = 3x^2 - 2xy + y^2$

a) $\partial_x f = 6x - 2y = \phi$, $\partial_y f = -2x + 2y = \phi$ → $\underline{y = x = \phi}$ → $\underline{f = \phi}$

$\partial_x^2 f = 6$, $\partial_y^2 f = 2$, $\partial_x \partial_y f = -2$ → $f_{xx} f_{yy} - f_{xy}^2 = 8 > \phi$ → $\underline{\text{Minimum}}$

b) $F = 3x^2 - 2xy + y^2 + \lambda(x^2 + y^2 + 1)$

$\left.\begin{array}{l}\partial_x F = 6x - 2y + 2\lambda x = \phi \\ \partial_y F = -2x + 2y + 2\lambda y = \phi\end{array}\right\}$ → $\underbrace{\begin{pmatrix} -3 & +1 \\ +1 & -1 \end{pmatrix}}_{\text{Hesse-Matrix von } f} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$

$\det \begin{pmatrix} -3-\lambda & 1 \\ 1 & -1-\lambda \end{pmatrix} = (3+\lambda)(1+\lambda) - 1 = \lambda^2 + 4\lambda + 2 \overset{!}{=} \phi$ → $\underline{\lambda_{1/2} = -2 \pm \sqrt{2}}$

λ_1: $\begin{pmatrix} -\sqrt{2} & 1 \\ 1 & 1-\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 \\ 1+\sqrt{2} \end{pmatrix} = \phi$ λ_2: $\begin{pmatrix} -1+\sqrt{2} & 1 \\ 1 & 1+\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 \\ 1-\sqrt{2} \end{pmatrix} = \phi$

↓ normieren ↓ normieren ↓

$\underline{\frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2+\sqrt{2}}} \\ \frac{1}{\sqrt{2-\sqrt{2}}} \end{pmatrix}}$ → $\underline{f = 2 - \sqrt{2}}$ $\underline{\frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2-\sqrt{2}}} \\ -\frac{1}{\sqrt{2+\sqrt{2}}} \end{pmatrix}}$ → $\underline{f = 2 + \sqrt{2}}$

c) → Computer

d) Globales Minimum $\underline{f = \phi}$
 — " — Maximum $\underline{f = 2 + \sqrt{2}}$

9) a) $\vec{\nabla} f = \begin{pmatrix} 4(x^2+y^2)x - y \\ 4(x^2+y^2)y - x \end{pmatrix}\Bigg|_{(x,y)=\vec{\phi}} = \begin{pmatrix} \phi \\ \phi \end{pmatrix}$ ✓ b) Spiegelung um 1. & 2. Mediane

c) $f(\vec{\phi}) = \phi$, $\vec{\nabla} f(\vec{\phi}) = \vec{\phi}$, $\partial_x^2 f = 4(x^2+y^2) + 8x^2 = \phi$, $\partial_x \partial_y f = -1$, $\partial_y^2 f = 4(x^2+y^2) + 8y^2 = \phi$

$\underline{f \approx -xy}$

d) $r^4 - r^2 \cos\phi \sin\phi = \phi$ → $r = \pm\sqrt{\cos\phi \sin\phi}$

7 a) $x^2+y^2=1$ & $3x+3y^2=4 \rightarrow 3x+3(1-x^2)=4 \rightarrow 3x^2-3x+1=\emptyset \rightarrow$
$\rightarrow x_{1,2}=(3\pm\sqrt{9-12})/6 \rightarrow x_{1/2}\in\mathbb{C}$

b & c) aufwändig: $f=(x_1-x_2)^2+(y_1-y_2)^2$
(bzw. stur rechnen) $\varphi_1=x_1^2+y_1^2-1$, $\varphi_2=3x_2+3y_2^2-4$
$F=f+\lambda_1\varphi_1+\lambda_2\varphi_2$

$$\boxed{\begin{array}{ll} \text{Gl 1)}\ \partial_{x_1}F=2(x_1-x_2)+2x_1\lambda_1=\emptyset & \text{Gl 5)}\ \varphi_1=\emptyset \\ \text{Gl 2)}\ \partial_{y_1}F=2(y_1-y_2)+2y_1\lambda_1=\emptyset & \text{Gl 6)}\ \varphi_2=\emptyset \\ \text{Gl 3)}\ \partial_{x_2}F=-2(x_1-x_2)+3\lambda_2=\emptyset & \\ \text{Gl 4)}\ \partial_{y_2}F=-2(y_1-y_2)+6y_2\lambda_2=\emptyset & \end{array}}$$

Gl 2 & Gl 4) $\rightarrow y_1\lambda_1+3y_2\lambda_2=\emptyset \rightarrow y_2/y_1=-\lambda_1/3\lambda_2$ Gl 2) $\rightarrow y_2/y_1=\lambda_1+1$

Gl 1 & Gl 3) $\rightarrow 3\lambda_2+2x_1\lambda_1=\emptyset \rightarrow x_1=-3/2\ \lambda_2/\lambda_1=1/2\ y_1/y_2$

Gl 1) $\rightarrow x_2=x_1(1+\lambda_1)=\frac{1}{2}\frac{y_1}{y_2}\frac{y_2}{y_1}=\underline{\underline{1/2}}$

Gl 6) $\rightarrow y_2=\pm\sqrt{4/3-x_2}=\underline{\underline{\pm\sqrt{5/6}}}$ Gl 5) $\rightarrow 1=x_1^2+y_1^2=\frac{1}{4}\frac{y_1^2}{y_2^2}+y_1^2 \rightarrow y_1=\frac{2y_2}{\sqrt{1+4y_2^2}}=\underline{\underline{\pm\sqrt{\frac{10}{13}}}}$

Gl 5) $\rightarrow x_1=\sqrt{1-y_1^2}=\underline{\underline{\pm\sqrt{3/13}}}$

b & c)
weniger aufwendig: $f=x^2+y^2$, $\varphi=3x+3y^2+4$
(bzw. vorher sich die Symmetrie überlegen) $F=f+\lambda\varphi=x^2+y^2+\lambda(3x+3y^2+4)$

$\left.\begin{array}{l}\partial_x F=2x+3\lambda=\emptyset \\ \partial_y F=2(1+3\lambda)y=\emptyset \rightarrow 3\lambda=-1\end{array}\right\} 2x=-3\lambda=1\rightarrow \underline{x=1/2}$

$3x+3y^2-4=\emptyset \rightarrow \underline{y=\pm\sqrt{5/6}}$

10) $\partial_t|\ (x^2+y^2)\sin(2t)-2xy=\emptyset \rightarrow 2(x\dot{x}+y\dot{y})\underbrace{\sin(2t)}_{=\emptyset}+2(\underbrace{x^2+y^2}_{=1})\underbrace{\cos(2t)}_{=-1}-$
$-2\dot{x}y-2x\dot{y}=\emptyset$
$\downarrow (x=\emptyset, y=1, t=\pi/2)$
$-2-2\dot{x}=\emptyset \rightarrow \underline{\dot{x}(t=\pi/2)=-1}$

$\partial_t|\ (x^2+y^2)\sin(t)-x\sin t-y=\emptyset \rightarrow 2(\underbrace{x}_{\emptyset}\dot{x}+\underbrace{y}_{1}\dot{y})\underbrace{\sin(t)}_{1}+(x^2+y^2)\underbrace{\cos(t)}_{\emptyset}-\underbrace{\dot{x}}_{-1}\underbrace{\sin t}_{1}+\underbrace{x\cos t}_{\emptyset}-\dot{y}=\emptyset$
$\downarrow (x=\emptyset, y=1, t=\pi/2)$
$\underline{\dot{y}(t=\pi/2)=-1}$