Analysis II für TPH, 4. Übung

1) $f = x+y+z$, $\varphi = xyz - 1 = 0$

$F = f + \lambda\varphi = x+y+z+\lambda(xyz-1)$

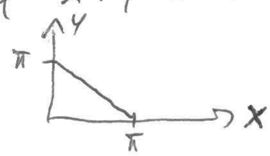
$\partial_x F = 1 + \lambda yz = 0$, $\partial_y F = 1 + \lambda xz = 0$, $\partial_z F = 1 + \lambda xy = 0$

$\rightarrow \lambda yz = \lambda xz = \lambda xy (= -1) \rightarrow x = y = z$

$xyz = 1 \rightarrow \underline{\underline{x = y = z = 1}} \rightarrow \underline{\underline{f = 3}}$

für $x = 1000$, $y = \frac{1}{1000}$, $z = 1 \rightarrow f = 1001 + \frac{1}{1000} > 3 \rightarrow f = 3$ ist ein <u>Minimum</u>

2) $f = \sin x \sin y$, $\varphi = x + y - \pi = 0$ $(x \geq 0, y \geq 0)$



$F = f + \lambda\varphi = \sin x \sin y + \lambda(x + y - \pi)$

$\partial_x F = \cos x \sin y + \lambda = 0$, $\partial_y F = \sin x \cos y + \lambda = 0$

$\Rightarrow \cos x \sin y = \sin x \cos y (= -\lambda)$ & $x + y = \pi$ erfüllt für

1) $x = y = \pi/2 \rightarrow f = 1$ Maximum

2) $x = 0, y = \pi \rightarrow f = 0$ Minima

3) $x = \pi, y = 0 \rightarrow f = 0$

3) (1) für $f \equiv 1 \rightarrow \|f\| = \max\limits_{0 \leq x \leq 1} |\underbrace{f'}_{0}| = 0$ aber $f \not\equiv 0 \rightarrow$ keine Norm

(2) o) $\|f\| = 0 = \max\limits_{0 \leq x \leq 1} (|f(x)| + |f'(x)|) \rightarrow |f(x)| = 0$ & $|f'(x)| = 0$ $\forall x \rightarrow \underline{\underline{f(x) \equiv 0}}$ ✓

o) $\|\lambda f\| = \max\limits_{0 \leq x \leq 1} (|\lambda f(x)| + |\lambda f'(x)|) = \lambda \max\limits_{0 \leq x \leq 1} (|f(x)| + |f'(x)|) = \underline{\underline{\lambda \|f\|}}$ ✓

o) $\|f + g\| = \max\limits_{0 \leq x \leq 1} \big(\underbrace{|f(x) + g(x)|}_{\leq |f(x)| + |g(x)|} + \underbrace{|(f(x) + g(x))'|}_{= |f'(x) + g'(x)| \leq |f'(x)| + |g'(x)|}\big) \leq \max\limits_{0 \leq x \leq 1} (|f(x)| + |f'(x)| + |g(x)| + |g'(x)|) \leq$

$\leq \max\limits_{0 \leq x \leq 1} (|f(x)| + |f'(x)|) + \max\limits_{0 \leq x \leq 1} (|g(x)| + |g'(x)|) = \underline{\underline{\|f\| + \|g\|}}$ ✓

$\Rightarrow$ <u>alle drei Bedingungen erfüllt</u>

4) a) $m \leq n$: $\|f_n - f_m\|_1 = \cancel{\int_{-2}^{-1/m} |-1-(-1)| \, dx} + \int_{-1/m}^{-1/n} |-1 - mx| \, dx + \int_{-1/n}^{+1/n} |nx - mx| \, dx +$

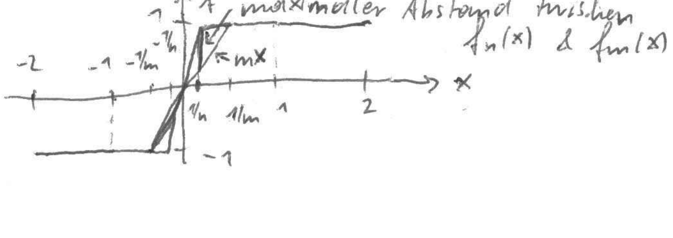
$+ \int_{1/n}^{1/m} |1 - mx| \, dx + \cancel{\int_{1/m}^{2} |1-1| \, dx} = 2 \int_{1/n}^{1/m} \underbrace{(1 - mx)}_{>0} \, dx + 2 \int_{0}^{1/n} \underbrace{(n-m)x}_{>0} \, dx =$

$= 2\left(\frac{1}{m} - \frac{1}{n}\right) - m\left(\frac{1}{m^2} - \frac{1}{n^2}\right) + (n-m)\frac{1}{n^2} = \underline{\frac{1}{m} - \frac{1}{n}} < \varepsilon$ $\forall n \geq m \geq 1/\varepsilon$ ✓

b) $m \leq n$ $\|f_n - f_m\|_\infty = \max\limits_{-2 \leq x \leq 2} |f_n(x) - f_m(x)| = 1 - m\frac{1}{n} \underset{\uparrow}{=} \frac{1}{2} \rightarrow$ für $\varepsilon < \frac{1}{2}$

für $n = 2m$

$\nexists N(\varepsilon)$ sodass $\forall n \geq m \geq N(\varepsilon)$ $\|f_n - f_m\|_\infty < \varepsilon$ ↯



5) $\|Kx\| \leq \|K\|\|x\|$ mit $\|K\|<1$

$x = Kx + b$, $x \overset{?}{=} \sum_{j=0}^{\infty} K^j b$

○) $\left.\begin{matrix} x = Kx+b \\ y = Ky+b \end{matrix}\right\} \to (y-x) = K(y-x) \to \|y-x\| = \|K(y-x)\| \leq \|K\|\|y-x\| \to$
$\to (1-\|K\|)\|y-x\| \leq 0 \to \|y-x\| = 0 \to y = x$ eindeutig

○) $\sum_{j=0}^{\infty} K^j b = K \sum_{j=0}^{\infty} K^j b + b = \sum_{j=1}^{\infty} K^j b + b = \sum_{j=0}^{\infty} K^j b$ ✓

○) konvergiert $\sum_{j=0}^{\infty} K^j b$?

$\left\| \sum_{j=0}^{\infty} K^j b \right\| \leq \sum_{j=0}^{\infty} \|K^j b\| \leq \sum_{j=0}^{\infty} \|K^j\|\|b\| = \sum_{j=0}^{\infty} \|K\|^j \|b\| \underset{0 \leq \|K\| < 1}{=} \|b\| \frac{1}{1-\|K\|}$ ✓

6)
○) $T(\alpha\{x_n\} + \beta\{y_n\}) = T(\{\alpha x_n + \beta y_n\}) = \left\{ \frac{\alpha x_n + \beta y_n}{n} \right\} =$

$= \alpha\{x_n/n\} + \beta\{y_n/n\} = \alpha T\{x_n\} + \beta T\{y_n\} \to$ linear

○) $\exists \{x_n\}$ sodass $T\{x_n\} = \{1\}$? $\to \left\{\frac{x_n}{n}\right\} = \{1\} \to$

$\to \{x_n\} = \{n\} \notin \ell^\infty \to$ nicht surjektiv

○) $\exists \{y_n\} \neq \{x_n\}$ sodass $T\{x_n\} = T\{y_n\}$? $\to \{x_n/n\} = \{y_n/n\} \to$

$\to x_n = y_n \; \forall n \to \{y_n\} = \{x_n\}$ ↯ $\to$ injektiv

○) $T^{-1}\{x_n\} = \{n x_n\} \to$ aber nur definiert auf Bildmenge $T(\ell^\infty) =$
$= \{\{y_n\} : \exists \{x_n\} \in \ell^\infty$ sodass $\{y_n\} = T\{x_n\}\}$

$T^{-1}\{1\} = \{n\} \notin \ell^\infty$

○) $T^{-1}(\alpha\{x_n\} + \beta\{y_n\}) =$ analog zu vorher $= \alpha T\{x_n\} + \beta T\{y_n\} \to$ linear

○) $\|T\{x_n\} - T\{y_n\}\|_\infty = \left\|\left\{\frac{x_n - y_n}{n}\right\}\right\|_\infty < \|\{x_n - y_n\}\|_\infty \to$ stetig

○) $a_0 = \{0, 0, \ldots\}$, $a_k = \{0, 0, \ldots, \underset{k\text{te Stelle}}{1}, 0, \ldots\}$

$\|a_0 - \alpha a_k\|_\infty = \alpha < \delta$, $\alpha \in \mathbb{R}$

$\|T^{-1}a_0 - T^{-1}\alpha a_k\|_\infty = \alpha k \to \nexists\, \delta(\varepsilon, a_0)$ sodass $\alpha k < \varepsilon \;\; \forall k \in \mathbb{N} \to$ nicht stetig

oder

$\|T^{-1} a_k\|_\infty = k = k\|a_k\|_\infty \to \left.\begin{matrix} T^{-1} \text{ ist nicht beschränkt} \\ T^{-1} \text{ ist linear} \end{matrix}\right\} \to$ nicht stetig

7) $0 \leq t \leq T < 1$

a) $\|Kx - Ky\|_\infty = \max_{0\leq t\leq T} |(Kx)(t) - (Ky)(t)| = \max_{0\leq t\leq T} \left| -3 \int_0^t s^2 (x(s) - y(s))\, ds \right| \leq$

$\leq \max_{0\leq t\leq T} 3 \int_0^t |s^2| \, |x(s) - y(s)|\, ds \leq \max_{0\leq t\leq T} 3 \int_0^t s^2 \|x-y\|_\infty\, ds =$

$= \max_{0\leq t\leq T} 3 \left.\frac{s^3}{3}\right|_0^t \|x-y\|_\infty = \max_{0\leq t\leq T} t^3 \|x-y\|_\infty = T^3 \|x-y\|_\infty \underset{T<1}{\leq} \|x-y\|_\infty$

b) $x_0(t) = 1$, $x_1(t) = 1 - 3\int_0^t s^2 x_0(s)\, ds = 1 - 3\int_0^t s^2\, ds = \underline{1 - t^3}$

$x_2(t) = 1 - 3\int_0^t s^2 x_1(s)\, ds = 1 - 3\int_0^t s^2 (1 - s^3)\, ds = \underline{1 - t^3 + \frac{1}{2} t^6}$

8) $\left(\begin{array}{l} \phi(x^*) = x^*, \quad \phi(y^*) = y^* \\ \|x^* - y^*\| = \|\phi(x^*) - \phi(y^*)\| \leq L\|x^* - y^*\| \rightarrow x^* = y^* \rightarrow \text{es gibt höchstens einen Fixpunkt} \end{array} \right)$

$\|x_n - x^*\| = \|\phi(x_{n-1}) - \phi(x^*)\| \leq \underline{L\|x_{n-1} - x^*\|} =$

$= L\|x_{n-1} - x_n + x_n - x^*\| \leq L\|x_{n-1} - x_n\| + L\|x_n - x^*\|$

$\rightarrow \|x_n - x^*\| \leq \underline{\frac{L}{1-L} \|x_{n-1} - x_n\|}$

$\|x_{n-1} - x_n\| = \|\phi(x_{n-2}) - \phi(x_{n-1})\| \leq L\|x_{n-2} - x_{n-1}\| \leq \ldots \leq L^{n-1} \|x_0 - x_1\|$

$\hookrightarrow \frac{L}{1-L}\|x_{n-1} - x_n\| \leq \underline{\frac{L^n}{1-L}\|x_0 - x_1\|}$

9) i) $\left.\begin{array}{l} |f(x)| = |x(t_0)| \leq \max_{t\in[a,b]} |x(t)| \\ x(t) = c = const: \ |f(x)| = |c| = \max_{t\in[a,b]} |c| = |c| \checkmark \end{array}\right\} \rightarrow \underline{\|f\| = 1}$

ii) $\left.\begin{array}{l} |f(x)| = \frac{1}{2}|x(a) + x(b)| \leq \max_{t\in[a,b]} |x(t)| \\ x(t) = c = const: \ |f(x)| = \frac{1}{2}|c + c| = |c| = \max_{t\in[a,b]} |c| = |c| \checkmark \end{array}\right\} \rightarrow \underline{\|f\| = 1}$

iii) $\left.\begin{array}{l} |f(x)| = \left|\int_a^b x(t)\, dt\right| \leq (b-a) \max_{t\in[a,b]} |x(t)| \\ x(t) = c = const: \ |f(x)| = \left|\int_a^b c\, dt\right| = (b-a)|c| = (b-a) \max_{t\in[a,b]} |x(t)| \checkmark \end{array}\right\} \rightarrow \underline{\|f\| = (b-a)}$

10) a) $x(t) = 0 = const$

$\|F(x)(t)\|_\infty = \max_t |B| = |B| \leq \|F\|_\infty \cdot \|x\|_\infty = \|F\|_\infty \cdot 0 \rightarrow \nexists \|F\|_\infty$

b) $\left.\begin{array}{l} \|F(x)(t)\|_\infty = \max_t \left|\int_a^t x(\tau)\, d\tau\right| \leq \max_t \int_a^t \|x\|_\infty\, d\tau = (b-a)\|x\|_\infty \\ x(t) = c = const \\ \|F(x)(t)\|_\infty = \max_t \int_a^t c\, d\tau = \max_t (t-a)c = (b-a)c = (b-a)\|x\|_\infty \end{array}\right\} \rightarrow \underline{\|F\|_\infty = (b-a)}$