UE Discrete Mathematics

Exercises for Oct 16/17, 2013

1) A simple undirected graph is called cubic if each of its vertices has degree 3.

(a) Find a cubic graph with 6 vertices!

(b) Is there a cubic graph with an odd number of vertices?

(c) Prove that for all $n \geq 2$ there exists a cubic graph with $2n$ vertices!

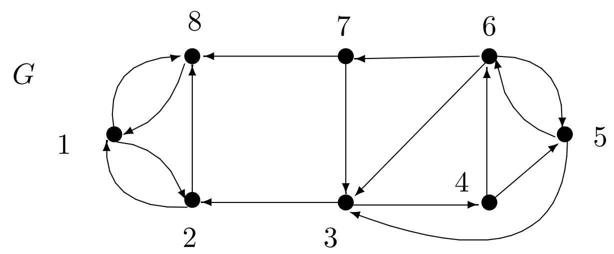
2) Use a suitable graph theoretical model to solve the following problems:

(a) Show that in every city at least two of its inhabitants have the same number of neighbours!

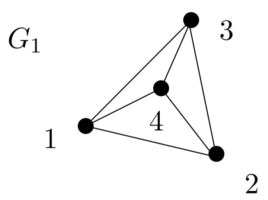
(b) 7 friends want to send postcards according to the following rules: (i) Each person sends and receives exactly 3 cards. (ii) Each one receives only cards from those to whom he or she sent a card and *vice versa*.

Tell how this can be done or prove that this is impossible!

3) Determine the adjacency matrix of the graph below and use it to compute the number of walks of length 3 from 6 to 4. Furthermore, compute the number of cycles of length 3 which contain the vertex 4.



4) Use the adjacency matrix of the following graph G_1 in order to compute the number of triangles (*i.e.* cycles of length 3) of G_1!



5) Show that each of the following four statements is equivalent to the statement „T is a tree“:

1. Every two nodes of T are connected by exactly one path.

2. T is connected and $\alpha_0(T) = \alpha_1(T) + 1$.

3. T is a minimal, connected graph, i.e., deleting an edge destroys connectivity.

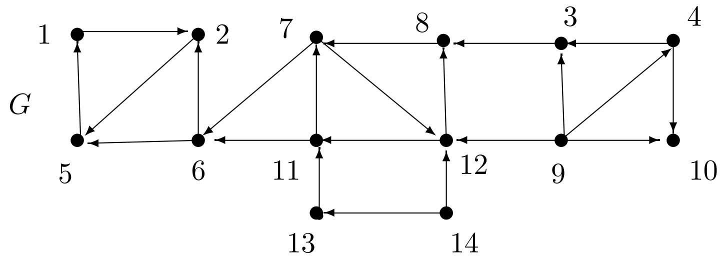
4. T is a maximal, acyclic graph, i.e., adding an edge generates a cycle.

6) Prove that a simple undirected graph G with n vertices and more than $(n-1)(n-2)/2$ is connected!

7) Let $G = (V, E)$ be a simple and undirected graph with $|V| > 4$. The complement $G^\kappa = (V^\kappa, E^\kappa)$ of G is defined as follows: $V^\kappa = V$ and $vw \in E^\kappa$ if and only if $vw \notin E$. Show that either G or G^κ (or both) must contain a cycle! Furthermore, determine all trees T such that T^κ is a tree as well!

8) Prove the following statements: If $G = (V, E)$ and $G' = (V', E')$ are isomorphic graphs and $\phi : V \to V'$ is an isomorphism, then $d_G(x) = d_{G'}(\phi(x))$ for all $x \in V$.
If, on the other hand, $\phi : V \to V'$ is a mapping satisfying $d_G(x) = d_{G'}(\phi(x))$ for all $x \in V$, then G and G' are not necessarily isomorphic.

9) Find the strongly connected components and the reduction G_R of the graph G below. Furthermore, determine all node bases of G.



10) Let $G = (V, E)$ be a simple and directed graph and G_R its reduction. Prove that G_R is acyclic!