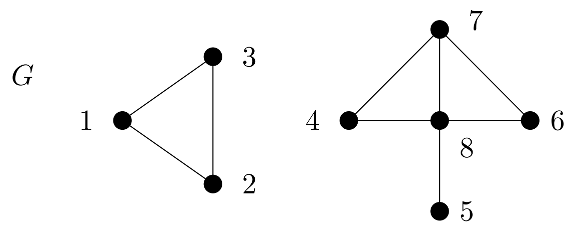
UE Discrete Mathematics

Exercises for Oct 23/24, 2013

11) Use the matrix tree theorem to compute the number of spanning forests of the graph below!



12) K_n denotes the complete graph with n vertices. Show that the number of spanning trees of K_n is n^{n-2}!

Hint: Use the matrix tree theorem and delete the first column and the first row of $D(K_n)-A(K_n)$. Then add all rows (except the first) to the first one and observe that all entries of the new first row are equal to 1. Use the new first row to transform the matrix in such a way that the submatrix built of the second to the last row and second to the last column is diagonal matrix.

13) If T is a tree having no vertex of degree 2, then T has more leaves than internal nodes. Prove this claim

(a) by induction,

(b) by considering the average degree and using the handshaking lemma.

14) Let $G=(V,E)$ be a connected graph with an even number of vertices. Show that there is a (not necessarily connected) spanning subgraph (i.e. a subgraph with vertex set V) in which all vertices have odd degree. Is this also true for non-connected graphs?

15) List all matroids (E,S) with $E=\{1\}$, $E=\{1,2\}$ or $E=\{1,2,3\}$.

16) Prove: If $M=(E,S)$ is a matroid and A and B are two bases of M, then $|A|=|B|$.

17) Let $G=(V,E)$ be an undirected graph. Set $M_k(G)=(E,S)$ where

$$S=\{A\subseteq E \mid A=F\cup M \text{ where } F \text{ is a forest and } |M|\le k\}.$$

Prove that $M_k(G)$ is a matroid.

18) Let $M=(E,S)$ be a matroid and $A\subseteq E$. The rank $r(A)$ of A is defined as the cardinality of a maximal independent subset of A. Prove that for all $A,B\subseteq E$ we have

(a) $r(A)\le |A|$, (b) $A\subseteq B$ implies $r(A)\le r(B)$, (c) $r(A\cap B)+r(A\cup B)\le r(A)+r(B)$.

19) Let $M=(E,S)$ be a matroid and $\mathcal{B}$ the family of all its bases. Let $A,B\in\mathcal{B}$ such that $A\neq B$. Prove that

(a) neither of the inclusions $A\subseteq B$ and $B\subseteq A$ holds,

(b) for each $x\in A$ there exists $y\in B$ such that $(A\setminus\{x\})\cup\{y\}\in\mathcal{B}$.

20) Let $\mathcal{B}$ be a family of sets which satisfies (a) and (b) of the previous exercise. Show that there is a matroid having $\mathcal{B}$ as its family of all its bases.