UE Discrete Mathematics

Exercises for Nov 13/14, 2013

31) Let G be an Eulerian graph and H be a subdivision of G. Is H Eulerian? Suppose that G is Hamiltonian. Does this imply that H is Hamiltonian as well?

32) For which m and n does the complete bipartite graph $K_{m,n}$ have a Hamiltonian cycle?

33–34) Use a suitable graph model to reformulate the exercises as graph theoretical problems.

33) Given a subset $A \subseteq \mathbb{R}^2$ which has area a and two decomposition of A into subsets $A_1, A_2 \ldots, A_m$ and $B_1, B_2, \ldots, B_m$ such that all the A_i's and all the B_i's have the same area a/m. Prove that there exists a permutation π of $\{1, 2, \ldots, m\}$ such that for all $i = 1, \ldots, m$ we have $A_i \cap B_{\pi(i)} \neq \emptyset$.

34) Given a set A with n elements and $B = \{A_1, A_2, \ldots, A_n\} \subseteq 2^A$. Prove that there exists an injective mapping $f : B \to A$ such that $f(A_i) \in A_i$ for all $i \in \{1, 2, \ldots, n\}$ if and only if for all $I \subseteq \{1, 2, \ldots, n\}$ the cardinality of $\bigcup_{i \in I} A_i$ is at least equal to the cardinality of I.

35) Follow the hint below to construct a schedule for the matches in a league of $2n$ teams which meets the following constraints:

(a) In each round each team plays exactly one match.

(b) In the end each team must have played against each of the other teams exactly once.

Hint: Consider the graph K_{2n} on the vertex set $\{1, 2, \ldots, 2n\}$ and show that each of the sets $M_i = \{1i\} \cup \{xy \mid x + y \equiv 2i \mod 2n - 1 \text{ and } x \neq y,\ x \neq 1,\ y \neq 1\}$ is a perfect matching (for $i = 2, \ldots, 2n$).

36) Let M be a matching of a simple and undirected graph $G = (V, W)$. An open path W in G is called *alternating* if exactly every other edge of W is in M. We call an alternating path *extending* if the start as well as the end vertex of W is not incident with any $e \in M$. Prove: If W is an extending alternating path, then $M \triangle W := (M \setminus W) \cup (W \setminus M)$ is a matching and $|M \triangle W| = |M| + 1$.

37) Let G be an undirected simple graph and H a subgraph of G satisfying $H \cong K_n$ for some n. What can be said about the relation between $\chi(G)$ and $\chi(H)$? Find a graph G with $\chi(G) = 3$ which does not have a K_3 as a subgraph.

38) Modify the proof of the Five Colour Theorem given in the lecture by replacing the induction hypothesis by "Assume that a simple planar graph with n vertices possesses a feasible colouring with 4 colours" to construct a "proof" of the Four Colour Theorem? Where is the flaw of this proof?

39) A block of a graph is a maximal (induced) subgraph which contains no cut vertex[1] and therefore no bridge as well. Every graph can be decomposed into blocks in a unique way and two different blocks have at most one vertex and no edge in common. The common vertex is always a cut vertex.
Given a connected graph G which has the blocks H_1, H_2,...,H_k. Assume that $\chi(H_i)$, $i = 1, 2, \ldots, k$, is known. Compute $\chi(G)$.

[1] A cut vertex is a vertex such that its removal increases the number of connected components. A bridge is an edge having this property. Thus the end vertices of a bridge are always cut vertices.

40) Show the following inequality for Ramsey numbers: If $r \geq 3$ then

$$R(n_1, \ldots, n_{r-2}, n_{r-1}, n_r) \leq R(n_1, \ldots, n_{r-2}, R(n_{r-1}, n_r))$$

Hint: Let $n = R(n_1, \ldots, n_{r-2}, R(n_{r-1}, n_r))$ and consider an edge colouring of K_n with r colours, say $c_1, \ldots, c_r$. Identify the colours c_{r-1} and c_r and apply the Ramsey property for $r-1$ colours.