UE Discrete Mathematics

Exercises for Nov 20/21, 2013

41) Prove that both, the sum and the product principle, can be extended to more than two sets, i.e. show that:

a) Gíven finite sets $A_1, A_2, \ldots, A_n$ which are pairwise disjoint, then

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i|.$$

b) Gíven finite sets $A_1, A_2, \ldots, A_n$, then

$$|A_1 \times A_2 \times \cdots \times A_n| = \prod_{i=1}^{n} |A_i|.$$

42) In how many ways can the letters a, a, b, b, c, d, e be listed such that the letter c and d are not in consecutive positions?

43) Let M be a non-empty set. Show that M has as many subsets with an odd number of elements as subsets with an even number of elements.

44) Find the number of ways to place n rooks on an $n \times n$ chess board such that no two of them attack each other.

45) Let n be a positive integer and let $(a_1, \ldots, a_n)$ be a permutation of $\{1, 2, \ldots, n\}$. Define

$$A_k = \{a_i \mid a_i < a_k, i > k\} \text{ and } B_k = \{a_i \mid a_i > a_k, i < k\}$$

for $1 \leq k \leq n$. Prove that $\sum_{k=1}^{n} |A_k| = \sum_{k=1}^{n} |B_k|$.

46) Let A be a set of 11 positive integers such that for all $x \in A$ we have $20 \not| x$. Prove that there are two integers $a, b \in A$ such that $20|(a+b)$ or $20|(a-b)$.

47) Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, \ldots, 82\}$. All points of the plane having coordinates (x, y) which satisfy $(x, y) \in A \times B$ are coloured with one of the colours red, green or blue. Prove that there exists a monochromatic rectangle.

Remark: A rectangle is called monochromatic if all its four vertices have the same colour.

48) Let $n \in \mathbb{N}$. Prove the identities

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \qquad \text{and} \qquad \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

by using only the combinatorial interpretation of the binomial coefficients.

49) Let $n \in \mathbb{N}$. Prove the identity

$$\sum_{m=0}^{n} \binom{m}{k} = \binom{n+1}{k+1}.$$

50) Prove that for all complex numbers x and all $k \in \mathbb{N}$ we have

$$\binom{-x}{k} = (-1)^k \binom{x+k-1}{k}.$$