UE Discrete Mathematics

Exercises for Nov 27/28, 2013

51) Compute the number of words made of $2n$ letters taken from the alphabet $\{a_1, a_2, \ldots, a_n\}$ such that each letter occurs exactly twice and no two consecutive letters are equal.

52) Let

$$f_n = |\{\pi \in S_n \mid \forall 1 \le i \le n \,:\, \pi(i) \ne i\}|.$$

Prove that $f_1 = 0$, $f_2 = 1$ and $f_n = (n-1)(f_{n-1} + f_{n-2})$. Furthermore, prove that this recurrence relation implies

$$f_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

53)

a) Find a simple expression for $S_{n,2}$.

b) Explain the identity $s_{n,n-1} = S_{n,n-1} = \binom{n}{2}$. (Find a simple reason, not a formal proof).

54) Prove the following identity:

$$x^n = \sum_{k=0}^{n} S_{n,k} (x)_k \qquad (n \ge 0).$$

55) Let A, B be two finite sets with $|A| = n$ and $|B| = k$. How many injective mappings $f : A \to B$ are there? Furthermore, show that the number of surjective mappings $f : A \to B$ equals $k! S_{n,k}$.

56) The n-th Bell number equals the number of set partitions of $\{1, 2, \ldots, n\}$. We set $B_0 := 1$. Prove the following identities:

$$B_n = \sum_{k=0}^{n} S_{n,k} \qquad \text{and} \qquad B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k.$$

57) Prove that the squares of the Fibonacci number satisfy the recurrence relation $a_{n+3} - 2a_{n+2} - 2a_{n+1} + a_n = 0$ and solve this recurrence relation with the correct initial conditions.

58) Let a_n denote the number of fat subsets of $\{1, 2, \ldots, n\}$ where a set A is called *fat* if $A = \emptyset$ or $\forall k \in A \,:\, k \ge |A|$. Prove that $a_n = F_{n+2}$ (as usual $(F_n)_{n \ge 0}$ denotes the sequence of the Fibonacci numbers) and show that this implies

$$F_{n+1} = \sum_{k=0}^{n} \binom{n-k}{k}.$$

59) Compute $\sum_{k=1}^{n-1} \frac{1}{k(n-k)}$ in two ways: (a) using a term by term partial fraction decomposition, (b) with generating functions.

60) Use generating functions to answer the following question: What is the number of solutions of the equation $a + b + c + d = 25$ if $a, b, c, d \in \{0, 1, 2, \ldots, 9\}$?