UE Discrete Mathematics

Exercises for Dec 11/12, 2013

71) Let $\mathcal{L}^{[d]}$ denote the set of words over the alphabet $\{a, b\}$ that contain exactly k occurrences of b such that there are always never more than d a's between two consecutive b's. Find a specification of $\mathcal{L}^{[d]}$ as combinatorial construction and use generating functions to compute the number of words in $\mathcal{L}^{[d]}$ having exactly n letters.

Remark: The result is an alternating sum which cannot be simplified further.

72) Use exponential generating functions to determine the number a_n of of ordered choices of n balls such that there are 2 or 4 red balls, an even number of green balls and an arbitrary number of blue balls.

73) An involution is a permutation π such that $\pi \circ \pi = \mathrm{id}_M$ where $M = \{1, 2, \ldots, n\}$. Let $\mathcal{I}$ be the set of involutions. Determine the exponential generating function $I(z)$ of $\mathcal{I}$.

74) Let $\mathcal{T}$ be the class of rooted and labelled trees, i.e. the n vertices of a tree of size n are labelled with the labels $1, 2, \ldots, n$. Use the theory of combinatorial constructions to determine a functional equation for the exponential generating function of $\mathcal{T}$. Finally, apply the following theorem to determine the number of trees with n vertices in $\mathcal{T}$. (You are not asked to prove the theorem.)

Theorem. *Let $\Phi(w) = \sum_{n\geq 0} \phi_n z^n$ with $\phi_0 \neq 0$. If $z = w/\phi(w)$, then $[z^n]w = \frac{1}{n}[w^{n-1}]\Phi(w)^n$.*

75) Let P be the set of all divisors of 12. Determine the Möbius function of $(P, |)$.

76) Let $(P, \leq)$ be the poset defined by $P = \{0, 1, 2, 3, 4\}$ and $0 \leq 1 \leq 4,\ 0 \leq 2 \leq 4,\ 0 \leq 3 \leq 4$. Compute all values $\mu(x, y)$ for $x, y \in P$.

77) Let $(P_1, \leq_1)$ and $(P_2, \leq_2)$ be two locally finite posets and $(P, \leq)$ be defined by $P = P_1 \times P_2$ and for $(a, x), (b, y) \in P$:

$$(a, x) \leq (b, y) \iff a \leq_1 b \wedge x \leq_2 y.$$

Show that $(P, \leq)$ is a poset and that the Möbius functions of P, P_1, P_2 satisfy $\mu_P((a, x), (b, y)) = \mu_{P_1}(a, b) \cdot \mu_{P_2}(x, y)$.

78) Let p, q, r be three distinct prime numbers and $m = pqr$. How many of the numbers $1, 2, \ldots, m$ are relatively prime to m? (Two numbers x and y are called relatively prime if their greatest common divisor is 1.)

79) Prove the following assertions:

(a) Every finite lattice has a 0-element and a 1-element.

(b) In every lattice L we have $(x \wedge y) \vee y = y$ for all $x, y \in L$

(c) There exists a lattice such that the following implication is not true:

$$x \leq z \implies \forall y : x \vee (y \wedge z) = (x \vee y) \wedge z.$$

80) Let $(P, \leq)$ be a finite poset. A subset $C \subseteq P$ is called a *chain* if $(C, \leq)$ is a linearly ordered set. A subset $A \subseteq P$ is called an *antichain* if no two elements of A are comparable with respect to $\leq$. A *chain cover* of P is a partition $P = C_1 \cup C_2 \cup \cdots \cup C_k$ in which all the C_i are chains. Dilworth's Theorem asserts that the size of any largest antichain is equal to the number of chains in a smallest chain cover.

Use Dilworth's Theorem to prove that every poset with at least $rs+1$ elements has either a chain with $r+1$ elements or an antichain with $s+1$ elements.