UE Discrete Mathematics

Exercises for Dec 18/19, 2013

81) Prove: If x and y are odd integers, then $2 \mid (x^2 + y^2)$ but $4 \nmid (x^2 + y^2)$.

82) Prove that for every integer n the number $n^2 - n$ is even and that $n^3 - n$ is a multiple of 6.

83) Let a, b, c, d be integers. Prove:

a) If $a \mid b$ and $a \mid c$, then for all integers x, y we have $a \mid (xb + yc)$.

b) If $\gcd(a, b) = 1$ and $c \mid a$ and $d \mid b$, then $\gcd(c, d) = 1$.

c) If $a \mid c$ and $b \mid c$ and $\gcd(a, b) = 1$, then $ab \mid c$

84) Prove that any two positive integers a, b satisfy $\gcd(a, b) \cdot \operatorname{lcm}(a, b) = a \cdot b$.

85) Use the Euklidean algorithm to find two integers a and b such that $2863a + 1057b = 42$.

86) Use the Euklidean algorithm to find all greatest common divisors of $x^3 + 5x^2 + 7x + 3$ and $x^3 + x^2 - 5x + 3$ in $\mathbb{Q}[x]$.

87) Let $(F_n)_{n \geq 0}$ be the Fibonacci sequence, i.e. $F_0 = 0, F_1 = 1, F_{n+1} = F_n + F_{n-1}$. Prove that any two consecutive Fibonacci numbers are coprime.

88) Prove that there exist infinitely many prime numbers p which are solutions of the equation $p \equiv 3 \mod 4$.

Hint: Assume that there are only finitely many such primes, say $p_1, \ldots, p_n$, and consider the number $4p_1p_2 \cdots p_n - 1$.

89) Which of the following mappings is well-defined?

a) $f : \mathbb{Z}_m \to \mathbb{Z}_m$, $\overline{x} \mapsto \overline{x^2}$

b) $g : \mathbb{Z}_m \to \mathbb{Z}_m$, $\overline{x} \mapsto \overline{2^x}$

90) Use the Chinese remainder theorem to solve the following system of congruence relations:

$$\begin{aligned} 3x &\equiv 12 \ (13) \\ 5x &\equiv 7 \ (22) \\ 2x &\equiv 3 \ (7) \end{aligned}$$