UE Discrete Mathematics

Exercises for Jan 15/16, 2014

101) List all irreducible polynomials up to degree 3 in $\mathbb{Z}_3$.

102) Decompose $x^4 + x^3 + 1$ into irreducible factors over $\mathbb{Z}_2$.

103) Let R be a ring and $(I_j)_{j\in J}$ be a family of ideals of R. Prove that $\bigcap_{j\in J} I_j$ is an ideal of R.

104) Let R be a ring and I be an ideal of R. Then $(R/I, +)$ is the factor group of $(R, +)$ over $(I, +)$. Define a multiplication on R/I by

$$(a + I) \cdot (b + I) := (ab) + I.$$

Prove that this operation is well-defined, *i.e.* that

$$\left.\begin{array}{rr} & a + I = c + I \\ \text{and} & b + I = d + I \end{array}\right\} \implies (ab) + I = (cd) + I.$$

Furthermore, show that $(R/I, +, \cdot)$ is a ring.

105) Show that $(\mathbb{Z}[x], +, \cdot)$ is a ring and that $1 \notin (\{x, x + 2\})$.

Remark: It can be shown that a principal ideal which is generated by $a_1, a_2, \dots, a_k$ can be alternatively generated by $\gcd(a_1, a_2, \dots, a_k)$. Therefore this example shows that $\mathbb{Z}[x]$ is a ring where not every ideal is a principal ideal. As a consequence, $\mathbb{Z}[x]$ cannot be a Euklidean ring.

106) Show that the set $\{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ with the usual addition and multiplication is a field. Compute $(3 - 5\sqrt{2})^{-1}$.

107) Show that the set $R = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$ with the usual addition and multiplication is an integral domain but not a field. Furthermore, prove that there are infinitely many units in R and give three concrete examples.

108) Give a reason why the Chinese remainder theorem (suitably modified) can be applied to the following system of congruence relations over $\mathbb{Q}[x]$ and use it to solve the system.

$$\begin{aligned}(x + 1)P(x) &\equiv 2x + 1 \ (x^2 + x + 1)\\ (x + 2)P(x) &\equiv 3x + 3 \ (x^2 + 2x + 3)\end{aligned}$$

109) Prove: If $(R, +, \cdot)$ is a ring and I_1, I_2 two of its ideals, then

$$\begin{aligned} I_1 + I_2 :=& \{a + b \mid a \in I_1, b \in I_2\} \text{ and}\\ I_1 * I_2 :=& \{a_1b_1 + a_2b_2 + \cdots + a_nb_n \mid n \geq 1, a_i \in I_1, b_i \in I_2 \text{ for } 1 \leq i \leq n\}\end{aligned}$$

are ideals as well.

110) Show that $\sqrt{2} + \sqrt{3}$ is algebraic over $\mathbb{Q}$ and determine its minimal polynomial.