VO Discrete Mathematics

6th Exercises for Nov 19/20, 2014

1) Use generating functions to find a closed form expressions for the sum $\sum_{k=0}^{n}(k^2+3k+2)$.

2) Compute

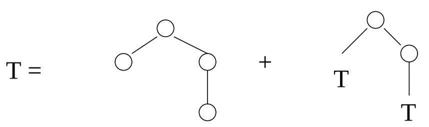
$$[z^n]\frac{2+3z^2}{\sqrt{1-5z}}.$$

3) Prove the following identity:

$$\sum_{n\geq 0}\binom{2n}{n}z^n = \frac{1}{\sqrt{1-4z}}.$$

4) A t-ary tree is a plane rooted tree such that every node has either t or 0 succesors. A node with t succesors is called internal nodes. How many leaves has a t-ary tree with n internal nodes? Moreover, let a_n be the number of t-ary trees with n internal nodes and $A(z)$ the generating function of this sequence. Find a functional equation for $A(z)$!

5) Compute the numbers t_n of plane rooted trees with n nodes which can be described by the equation



6) Compute the number of plane rooted trees with n nodes.

7) Consider a regular $(n+2)$-gon A, say, with the vertices $0,1\ldots,n+1$. A triangulation is a decomposition of A into n triangles such that the 3 vertices of each triangle are vertices of A as well. Show that the set $\mathcal{T}$ of triangulations of regular polygons can be described as a combinatorial construction satisfying

$$\mathcal{T} = \{\varepsilon\} \cup \mathcal{T} \times \Delta \times \mathcal{T}$$

where Δ denotes a single triangle **and ε denotes the empty triangulation (consisting of no triangle and corresponding to the case $n=0$)**. What is the number of triangulations of A?

8) Let $\mathcal{L}$ denote the set of words over the alphabet $\{a,b\}$ that contain exactly k occurrences of b. Obviously, the number of words in $\mathcal{L}$ which have exactly n letters is $\binom{n}{k}$. Prove this by finding a specification of $\mathcal{L}$ as combinatorial construction and translating this specification into generating functions.

9) Let $\mathcal{L}^{[d]}$ denote the set of words over the alphabet $\{a,b\}$ that contain exactly k occurrences of b such that there are always never more than d a's between two consecutive b's. Find a specification of $\mathcal{L}^{[d]}$ as combinatorial construction and use generating functions to compute the number of words in $\mathcal{L}^{[d]}$ having exactly n letters.
Remark: The result is an alternating sum which cannot be simplified further.

10) Let s_{nk} be the Stirling numbers of the first kind, that is, the number of permutations of $\{1, 2, \ldots, n\}$, where the cycle representation has exactly k cycles. Prove

$$\sum_{n,k} s_{nk} \frac{z^n}{n!} u^k = e^{u \log \frac{1}{1-z}} = \frac{1}{(1-z)^u}.$$

Remark: Start with the identity

$$\sum_{k=0}^{n} s_{nk} u^k = u(u+1)(u+2)\ldots(u+n-1).$$