1. Exercise sheet - Analysis on Manifolds - 2021

1.1 Topological Manifolds

1. Let X be a locally path-connected topology space. Show that
   (a) the connected components of X are open in X,
   (b) the path-connected components of X are the same its connected components, and
   (c) X is connected if and only if X is path-connected.

2. Let X be a topological space.
   (a) Suppose that f, f' and g, g' are paths in X with the same start and end points as well as $f(1) = f'(1) = g(0) = g'(0)$. Show that if $f \sim f'$ and $g \sim g'$, then $f \cdot g \sim f' \cdot g'$.
   (b) Show that for paths f, g, h in X,
   $$([f] \cdot [g]) \cdot [h] = [f] \cdot ([g] \cdot [h]),$$
   whenever the products are defined.

3. Let X be a path-connected topological space. Prove the following statements.
   (a) The fundamental groups of X at different base-points are all isomorphic.
   (b) X is simply connected if and only if every two paths in X, with the same initial and end points, are homotopic.

4. Let $M_1, \ldots, M_k$ be topological manifolds of dimensions $n_1, \ldots, n_k$, respectively. Show that $M_1 \times \cdots \times M_k$ is a topological manifold of dimension $n_1 + \cdots + n_k$, with charts of the form $(U_1 \times \cdots \times U_k, \varphi_1 \times \cdots \times \varphi_k)$, where (U_i, φ_i) is a chart of M_i for all $1 \leq i \leq k$.

5. Show that the defining properties of topological Manifold are independent of each other by giving examples of topological spaces each of which has exactly two of the three defining properties.

6. Let M be a topological manifold, and let $\mathcal{U}$ be an open cover of M.
   (a) Assuming that each set in $\mathcal{U}$ intersects only finitely many others, show that $\mathcal{U}$ is locally finite.
   (b) Give an example to show that the converse (a) is false in general.
   (c) Now assume the sets in $\mathcal{U}$ are precompact in M and prove the converse of (a).

7. Suppose that M is a locally Euclidean Hausdorff space of dimension $n \geq 1$. Show that M is second-countable if and only if it is paracompact and has countably many connected components.

1.2 Smooth Structures

8. Show that if a non-empty topological manifold M of dimension $n \geq 1$ admits a smooth structure, then it admits infinitely many of them.

9. Denote by $N := (0, \ldots, 0, 1)$ the "north pole" of the sphere $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ and $S := -N$ the "south pole". The *Stereographic projection* $\sigma : \mathbb{S}^n \backslash \{N\} \to \mathbb{R}^n$ is defined by

$$\sigma(x^1, \ldots, x^{n+1}) = \frac{(x^1, \ldots, x^n)}{1 - x^{n+1}}.$$

For $x \in \mathbb{S}^n \backslash \{S\}$ define $\bar{\sigma}(x) = -\sigma(-x)$. Show that

(a) σ is bijective and its inverse is given by

$$\sigma^{-1}(u^1, \ldots, u^n) = \frac{(2u^1, \ldots, 2u^n, \|u\|^2 - 1)}{\|u\|^2 + 1},$$

(b) the atlas of $\mathbb{S}^n$ consisting of the two charts $(\mathbb{S}^n \backslash \{N\}, \sigma)$ and $(\mathbb{S}^n \backslash \{S\}, \bar{\sigma})$ define a smooth structure on $\mathbb{S}^n$, and

(c) the smooth structure defined in (b) is the same as the standard smooth structure of $\mathbb{S}^n$ defined in the lectures.

10. By identifying $\mathbb{R}^2$ with $\mathbb{C}$, we can think of $\mathbb{S}^1$ as a subset of the complex plane. An angular function on a subset U of $\mathbb{S}^1$ is a continuous function $\theta : U \to \mathbb{R}$ such that $e^{i\theta(z)} = z$ for all $z \in U$. Show that there exists an angular function θ on an open subset $U \subset \mathbb{S}^1$ if and only if $U \neq \mathbb{S}^1$. For any such angular function show that (U, θ) is a smooth coordinate chart for $\mathbb{S}^1$ with its standard smooth structure.

11. Let $M = \mathrm{cl}\,\mathbb{B}^n$ be the closure of the unit Euclidean ball in $\mathbb{R}^n$. Show that M is a topological manifold with boundary for which each point in $\mathbb{S}^{n-1}$ is a boundary point, and each point in $\mathbb{B}^n$ is an interior point. Show how to give a smooth structure to M such that every smooth interior chart is a smooth chart for the standard smooth structure on $\mathbb{B}^n$.