2. Exercise sheet - Analysis on Manifolds - 2021

2.1 Smooth Maps

1. Define $f : \mathbb{R} \to \mathbb{R}$ by
$$f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0. \end{cases}$$
Show that for every $x \in \mathbb{R}$, there are smooth coordinate charts (U, φ) containing x and (V, ψ) containing $f(x)$ such that $\psi \circ f \circ \varphi^{-1}$ is a smooth map from $\varphi(U \cap f^{-1}(V))$ to $\phi(V)$ but f is not smooth as a map between manifolds.

2. Suppose that $F : M \to N$ is a smooth map between smooth manifolds with or without boundary. Show that the coordinate representation of F with respect to any pair of smooth charts for M and N is smooth.

3. Show that the inclusion map $\mathrm{cl}(\mathbb{B}^n) \hookrightarrow \mathbb{R}^n$ is smooth when $\mathrm{cl}(\mathbb{B}^n)$ is regarded as a smooth manifold with boundary.

4. The n-dimensional projective space $\mathbb{RP}^n$ is the set of all 1-dimensional subspaces of $\mathbb{R}^{n+1}$ endowed $\mathbb{RP}^n$ with the quotient topology determined by the map $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ sending each point $x \in \mathbb{R}^{n+1} \setminus \{0\}$ to the the subspace spanned by x denoted by $[x]$.

   For each $i = 1, \ldots, n+1$, let $\tilde{U}_i = \{x \in \mathbb{R}^{n+1} \setminus \{0\} : x_i \neq 0\}$ and $U_i = \pi(\tilde{U}_i) \subseteq \mathbb{RP}^n$. Define $\varphi_i : U_i \to \mathbb{R}^n$ by
$$\varphi_i[x^1, \ldots, x^{n+1}] = \left(\frac{x^1}{x^i}, \ldots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i} \cdots \frac{x^{n+1}}{x^i} \right).$$
   Show that:

   (a) U_i is open.

   (b) ϕ_i is well defined.

   (c) ϕ_i a homeomorphism.

   (d) $\mathbb{RP}^n$ is a n-dimensional topological manifold.

   (e) The coordinate charts (φ_i, U_i) are smoothly compatible (they define a smooth structure on $\mathbb{RP}^n$).

   (f) If $d \in \mathbb{Z}$ and $P : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^{n+1} \setminus \{0\}$ is a smooth map such that $P(\lambda x) = \lambda^d P(x)$ for all $x \in \mathbb{R}^{n+1} \setminus \{0\}$ and $\lambda \in \mathbb{R} \setminus \{0\}$, then the map $\bar{P} : \mathbb{RP}^n \to \mathbb{RP}^n$ defined by $\bar{P}([x]) = [P(x)]$ is well defined and smooth with respect to the smooth structure defined in (e).

5. Suppose that M is a topological space with the property that for every indexed open cover $\mathcal{X}$ of M, there exists a partition of the unity subordinate to $\mathcal{X}$. Show that M is paracompact.

6. Suppose that A and B are disjoint closed subset of a smooth manifold M. Show that there exists $f \in C^\infty(M)$ such that $0 \leq f(x) \leq 1$ for all $x \in M$, $f^{-1}(0) = A$, and $f^{-1}(1) = B$.

7. Suppose that $M_1, \ldots, M_k, N$ are smooth manifolds and let $\pi_i : M_1 \times \cdots \times M_k \to M_i$ be the projection onto the i-th factor. Show that a map $F : N \to M_1 \times \cdots \times M_k$ is smooth if and only if all its component functions $\pi_i \circ F : N \to M_i$ are smooth.

1.2 Tangent Vectors

8. Suppose that $M_1, \ldots, M_k$ are smooth manifolds, and for each j, let $\pi_j : M_1 \times \cdots \times M_k \to M_j$ be the projection onto the M_j factor. Show that for any point $p = (p_1, \ldots, p_k) \in M_1 \times \cdots \times M_k$, the map

$$\alpha : T_p(M_1 \times \cdots \times M_k) \to T_{p_1}M_1 \times \cdots \times T_{p_k}M_k$$

defined by

$$\alpha(v) = (d(\pi_1)_p(v), \ldots, d(\pi_k)_p(v))$$

is an isomorphism.

9. Prove that if M and N are smooth manifolds, then $T(M \times N)$ is diffeomorphic to $TM \times TN$.

10. Show that $T\mathbb{S}^1$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}$.

11. Let M,N, and P be smooth manifold, let $F : M \to N$ and $G : N \to P$ be smooth maps, and $p \in M$. Show that:

    (a) $dF_p : T_pM \to T_{F(p)}N$ is linear.

    (b) $d(G \circ F)_p = dG_{F(p)} \circ dF_p : T_pM \to T_{G \circ F(p)}P$.

    (b) $d(Id_M)_p = Id_{T_pM} : T_pM \to T_pM$.

    (d) If F is a diffeomorphism, then dF_p is an isomorphism, and $(dF_p)^{-1} = d(F^{-1})_{F(p)}$.

    (e) If M is connected and $dF_p : T_pM \to T_{F(p)}N$ is the zero map for all $p \in M$, then F is constant.

12. Let M be a smooth manifold and p be a point of M. Let $C^\infty(M)$ denote the lgera of germs of smooth real valued functions at p, and let $\mathcal{D}_pM$ denote the vector space of derivations of $C_p^\infty(M)$. Define the map $\Phi : \mathcal{D}_pM \to T_pM$ by $(\Phi v)f = v([f]_p)$. Show that Φ is an isomorphism.

13. Let M be a smooth manifold and $p \in M$. Let $\mathcal{V}_p$ denote the set of equivalence classes of smooth curves starting at p under the relation $\gamma_1 \sim \gamma_2$ if $(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$ for every smooth real-valued function f defined in a neighborhood of p. Show that the map $\Psi : \mathcal{V}_p M \to T_p M$ defined by $\Psi[\gamma] = \gamma'(0)$ is well defined and bijective.