3. Exercise sheet - Analysis on Manifolds - 2021

1. Suppose that M and N are smooth manifolds and $f : M \to N$ is a smooth map. Show that F is a local diffeomorphism if and only if F is a immersion and a submersion.

2. Use the inclusion map $\mathbb{H}^n \hookrightarrow \mathbb{R}^n$ to show that the inverse function theorem for manifolds does not extend to the case in which the domain of the smooth map in question is a smooth manifold with boundary.

3. Suppose M is a smooth manifold (with boundary), N is a smooth manifold with boundary, and $F : M \to N$ is smooth. Show that if $p \in M$ is a point such that dF_p is nonsingular then $F(p) \in \operatorname{int} N$

4. Let M be a nonempty compact manifold. Show that there is no smooth submersion $F : M \to \mathbb{R}^k$ for any $k > 0$.

5. Let M be a connected smooth manifold and let $\pi : E \to M$ be a topological covering map. Show that there is only one smooth structure on E such that π is a smooth covering map. *Hint: use the existence of smooth local sections.*

6. Show that the map $\pi : \mathbb{R}^n \setminus \{0\} \to \mathbb{RP}^n$ that is used to define the projective space is smooth (exercise sheet 2, exercise 4). Show that its restriction to $\mathbb{S}^n$ is a two-sheeted smooth covering map.

7. Consider $\mathbb{S}^1$ with its standard smooth structure and let $\varepsilon : \mathbb{R} \to \mathbb{S}^1$ be defined by $\varepsilon(t) = e^{2\pi i t}$. Show that:

   - the coordinate representation of ε with respect to any angle coordinate θ for $\mathbb{S}^1$ is of the form $\hat{\varepsilon}(t) = 2\pi t + c$ for some constant c,
   - the map $\varepsilon^n : \mathbb{R}^n \to \mathbb{T}^n$ defined by $\varepsilon^n(x^1, \dots, x^n) = (e^{2\pi i x^1}, \dots, e^{2\pi i x^n})$ is a smooth covering map for all $n \geq 1$,
   - the map $X : \mathbb{R}^2 \to \mathbb{R}^3$ given by

     $$X(u, v) = ((2 + \cos 2\pi u) \cos 2\pi v, (2 + \cos 2\pi u) sin 2\pi v, \sin 2\pi u)$$

     is a smooth immersion of $\mathbb{R}^2$ into $\mathbb{R}^3$ whose image is the torus of revolution obtain by revolving the circle $(y - 2)^2 + z^2 = 1$ in the (y, z)-plane about the z-axis.
   - Using ε^2, show that the immersion X descends to a smooth embedding of $\mathbb{T}^2$ into $\mathbb{R}^3$. Specifically, show that X passes to the quotient to define a smooth map $\widetilde{X} : \mathbb{T}^2 \to \mathbb{R}^3$, and then show that $\widetilde{X}$ is a smooth embedding whose image is the given surface of revolution.

8. Define the map $F : \mathbb{S}^2 \to \mathbb{R}^4$ by $F(x, y, z) = (x^2 - y^2, xy, xz, yz)$. Using the smooth covering map of exercise 6, show that F descends to a smooth embedding of $\mathbb{RP}^2$ into $\mathbb{R}^4$.

9. Define the map $\Phi : \mathbb{R}^4 \to \mathbb{R}^2$ by

$$\Phi(x, y, s, t) = (x^2 + y, x^2 + y^2 + s^2 + t^2 + y).$$

Show that $(0, 1)$ is a regular value of Φ, and that the level set $\Phi^{-1}(0, 1)$ is diffeomorphic to $\mathbb{S}^2$

10. Let M be a smooth n-manifold. Suppose that $S \subset M$ is such that for each $p \in S$ there is a neighborhood $U \subset M$ such that $U \cap S$ is an embedded k-submanifold of U. Show that S is an embedded k-submanifold of M.

11. Let M be a smooth n-manifold with a boundary. Show that ∂M is a embedded $(n-1)$-submanifold (without a boundary) of M.