2. Exercise sheet - Analysis on Manifolds - 2021

1. Suppose that $M \subseteq \mathbb{R}^n$ is an embedded m-dimensional submanifold, and let $UM \subseteq T\mathbb{R}^n$ be the set of all unit tangent vectors to M:

$$UM = \{(x, v) \in T\mathbb{R}^n : x \in M,\ v \in T_xM,\ |v| = 1\}.$$

It is called the unit tangent bundle of M. Prove that UM is an embedded $(2m-1)$-dimensional submanifold of $T\mathbb{R}^n \approx \mathbb{R}^n \times \mathbb{R}^n$.

2. For each $a \in \mathbb{R}$, let M_a be the subset of $\mathbb{R}^2$ defined by

$$M_a = \{(x, y) : y^2 = x(x-1)(x-a)\}.$$

For which values of a is M_a an embedded sunmanifold of $\mathbb{R}^2$? For which values can M_a be a topology and smooth structure making it into a immersed submanifold?

3. The vector field V on $\mathbb{R}^n$, whose value at $x \in \mathbb{R}^n$ is

$$V_x = x^1 \left.\frac{\partial}{\partial x^1}\right|_x + \cdots + x^n \left.\frac{\partial}{\partial x^n}\right|_x,$$

is called the Euler vector field.

   (a) Show that V is smooth.

   (b) Let c be a real number, and $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be a smooth function that is positively homogeneous of degree c, meaning that $f(\lambda x) = \lambda^c f(x)$ fro all $\lambda > 0$ and $x \in \mathbb{R}^n \setminus \{0\}$. Prove that $Vf = cf$.

4. Let M be a nonempty positive-dimensional smooth manifold. Show that $\mathfrak{X}(M)$ is infinite dimensional.

5. Let M be a smooth manifold with boundary. Show that there exists a global smooth vector field on M whose restriction to ∂M is everywhere inward-pointing, and one whose restriction to ∂M is everywhere outward-pointing.

6. Let M be the open submanifold of $\mathbb{R}^2$ where x and y are positive, and let $F : M \to M$ be the map $F(x, y) = (xy, y/x)$. Show that F is a diffeomorphim, and compute F_*X and F_*Y, where

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad Y = y\frac{\partial}{\partial x}.$$

7. for each of the following vector fields on the plane, compute its coordinate representation in polar coordinates on the right half plane $\{(x,y) : x > 0\}$.

   (a) $X = x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y}$.

   (b) $Y = x\dfrac{\partial}{\partial y} - y\dfrac{\partial}{\partial x}$.

   (c) $Z = (x^2 + y^2)\dfrac{\partial}{\partial x}$.

8. Let $F : \mathbb{R}^2 \to \mathbb{RP}^2$ be the smooth map $F(x,y) = [x,y,1]$, and let $X \in \mathfrak{X}(\mathbb{R}^2)$ be defined by $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$. Prove that there is a vector field $Y \in \mathfrak{X}(\mathbb{RP}^2)$ that is F-related to X, and compute its coordinate representation in terms of the charts defined in exercise 4 exercise sheet 2.

9. Show that there is a smooth vector field on $\mathbb{S}^2$ that vanishes at exactly one point.

10. Let M be a smooth manifold with or without boundary, let N be a smooth manifold, and let $f : M \to N$ be a smooth map. Define $F : M \to M \times N$ by $F(x) = (x, f(x))$. Show that for every $X \in \mathfrak{X}(M)$, there s a smooth vector field on $M \times N$ that is F-related to X.

11. Let M be a smooth manifold, $f \in C^\infty(M)$ and $Y \in \mathfrak{X}(M)$. Show that $fY : M \to TM$ defined by $(fY)_p = f(p)Y_p$ is a smooth vector field.

12. Let M be a smooth manifold, $f \in C^\infty(M)$ and $Y \in \mathfrak{X}(M)$. Show that $fY : M \to TM$ defined by $(fY)_p = f(p)Y_p$ is a smooth vector field.

13. Suppose that M is a smooth manifold and $S \subseteq M$ is an embedded submanifold with or without boundary. Given $X \in \mathfrak{X}(S)$, show that there is a smooth vector field Y on a neighborhood of S in M such that $X = Y|_S$ Show that every such vector field extends to all of M if and only if S is properly embedded.