5. Exercise sheet - Analysis on Manifolds - 2021

1. Suppose M is a smooth n-manifold, $p \in M$; and $y^1 \dots, y^k$ are smooth real-valued functions defined on a neighborhood of p in M. Prove the following statements.

   (a) If $k = n$ and $(dy^1|_p, \dots, dy^n|_p)$ is a basis for T_pM; then $(y^1, \dots y^n)$ are smooth coordinates for M in some neighborhood of p.

   (b) If $(dy^1|_p, \dots, dy^k|_p)$ is a linearly independent k-tuple of covectors and $k < n$, then there are smooth functions $y^{k+1}, \dots, y^n$ such that $(y^1, \dots y^n)$ are smooth coordinates for M in a neighborhood of p.

   (c) If $(dy^1|_p, \dots, dy^k|_p)$ span T^*_pM; there are indices $i_1, \dots, i_n$ such that $(y^{i_1}, \dots, y^{i_n})$ are smooth coordinates for M in a neighborhood of p

2. Let M be a smooth manifold, and $C \subset M$ be an embedded submanifold. Let $f \in C^\infty(M)$, and suppose $p \in C$ is a point at which f attains a local maximum or minimum value among points in C. Given a smooth local defining function $\Phi : U \to \mathbb{R}^k$ for C on a neighborhood U of p in M; show that there are real number $\lambda_1, \dots, \lambda_k$ (called Lagrange multipliers) such that

$$df_p = \lambda_1 d\Phi^1|_p + \dots + \lambda_k d\Phi^k|_p$$

3. Show that any two points in a connected smooth manifold can be joined by a smooth curve segment.

4. The *length* of a smooth curve segment $\gamma : [a, b] \to \mathbb{R}^n$ is defined by the value of the (ordinary) integral

$$\mathrm{L}(\gamma) = \int_a^b |\gamma'(t)| dt.$$

   show that there is no smooth covector $\omega \in \mathfrak{X}^*(\mathbb{R}^n)$ with the property that $\int_\gamma \omega = \mathrm{L}(\gamma)$ for every smooth curve γ.

5. Let M be a compact manifold of positive dimension. Show that every exact covector field on M vanishes at least at two points in each component of M.

6. Compute the flow of each of the following vector fields on $\mathbb{R}^2$:

   (a) $V = y\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y}$.

   (b) $W = x\dfrac{\partial}{\partial x} + 2y\dfrac{\partial}{\partial y}$.

(c) $X = x\dfrac{\partial}{\partial x} - y\dfrac{\partial}{\partial y}$.

7. Suppose M is a smooth, compact manifold that admits a nowhere vanishing smooth vector field. Show that there exists a smooth map $F : M \to M$ that is homotopic to the identity and has no fixed points.

8. Let M be a connected smooth manifold. Show that the group of diffeomorphisms of M acts transitively on M: that is, for any $p, q \in M$; there is a diffeomorphism $F : M \to M$ such that $F(p) = q$. *Hint: first prove that if $p, q \in \mathbb{B}^n$ (the open unit ball in $\mathbb{R}^n$), there is a compactly supported smooth vector field on $\mathbb{B}^n$ whose flow θ satisfies $\theta_1(p) = q$.*

9. Let M be a smooth manifold and let $S \subseteq M$ be a compact embedded submanifold. Suppose $V \in \mathfrak{X}(M)$ is a smooth vector field that is nowhere tangent to S. Show that there exists $\varepsilon > 0$ such that the flow of V restricts to a smooth embedding $\Phi : (-\varepsilon, \varepsilon) \times S \to M$.

10. Give an example of finite-dimensional vector spaces V and W and a specific element $\alpha \in V \otimes W$ that cannot be expressed as $v \otimes w$ for $v \in V$ and $w \in W$.

11. Let V be an n-dimensional real vector space. Show that

$$\dim \Sigma^k(V^*) = \binom{n+k-1}{k} = \frac{(n+k-1)!}{k!(n-1)!}.$$

12. Prove the following statements:

    (a) The symmetrical product is commutative and bilinear.

    (b) If S, T are covectors, then

$$ST = \frac{S \otimes T + T \otimes S}{2}.$$