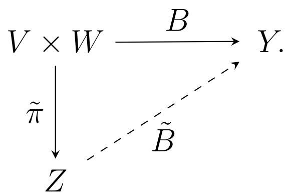
6. Exercise sheet - Analysis on Manifolds - 2021

1. Suppose that $\tilde{\pi} : V \times W \to Z$ is a bilinear map into a vector space Z with the following property: for any bilinear map $B : V \times W \to Y$, there is a unique linear map $\tilde{B} : Z \to Y$ such that the following diagram commutes:



   Then there exists a unique isomorphism $\Phi : V \otimes W \to Z$ such that $\tilde{\pi} = \Phi \circ \pi$, where π is the canonical projection $\pi : V \times W \to V \otimes W$. [Remark: this shows that the details of the construction used to define the tensor product space are irrelevant, as long as the resulting space satisfies the characteristic property.]

2. Let $V_1, \ldots, V_k$ and W be finite-dimensional real vector spaces. Prove that there is a canonical (basis-independent) isomorphism

$$V_1^* \otimes \cdots \otimes V_k^* \otimes W \cong \mathrm{L}(V_1, \ldots, V_k; W).$$

3. Show that covectors $\omega^1, \cdots, \omega^k$ on a finite-dimensional vector space are linearly dependent if and only if $\omega^1 \wedge \cdots \wedge \omega^k = 0$.

4. Let M be a smooth n-manifold with or without boundary, and let $(\omega^1, \ldots, \omega^k)$ be an ordered k-tuple of smooth 1-forms on an open subset $U \subset M$ such that $(\omega^1|_p, \ldots, \omega^k|_p)$ is linearly independent for each $p \in U$. Given smooth 1-forms $\alpha^1, \ldots, \alpha^k$ on U such that

$$\sum_{i=1}^{k} \alpha^i \wedge w^i = 0,$$

   show that each α^i can be written as a linear combination of $\omega^1 \ldots \omega^k$ with smooth coefficients.

5. Define a 2-form ω on $\mathbb{R}^3$ by

$$w = x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy$$

   (a) Compute ω in spherical coordinates (ρ, φ, θ) defined by

$$(x, y, z) = (\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho \cos\varphi).$$

(b) Compute $d\omega$ in both Cartesian and spherical coordinates and verify that both expressions represent the same 3-form.

(c) Compute the pullback $\iota_{\mathbb{S}^2}^*\omega$ to $\mathbb{S}^2$, using coordinates (ϕ, θ) on the open subset where these coordinates are defined.

(b) Show that $\iota_{\mathbb{S}^2}^*\omega$ is not zero.

6. Suppose M is a smooth manifold that is the union of two orientable open submanifolds with connected intersection. Show that M is orientable. Use this to give another proof that $\mathbb{S}^n$ is orientable.

7. Suppose M and N are oriented smooth manifolds with or without boundary, and $F : M \to N$ is a local diffeomorphism. Show that if M is connected, then F is either orientation-preserving or orientation-reversing

8. Let θ be a smooth flow on an oriented smooth manifold with or without boundary. Show that for each $t \in \mathbb{R}$, θ_t is orientation-preserving wherever it is defined.

9. Let $v_1 \ldots, v_n$ be any n linearly independent vectors in $\mathbb{R}^n$, and let P be the n-dimensional parallelepiped they span:

$$P = \{t_1 v_1 + \ldots t_n v_n : 0 \geq t_i \geq 1\}.$$

Show that $\mathrm{Vol}(P) = |\det(v_1, \ldots, v_n)|$.

10. Let $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1 \subseteq \mathbb{R}^4$ denote the 2-torus, defined as the set of points (w, x, y, z) such that $w^2 + x^2 = y^2 + z^2 = 1$, with the product orientation determined by the standard orientation on $\mathbb{S}^1$. Compute $\int_{\mathbb{T}^2} \omega$, where ω is the following 2-form on $\mathbb{R}^4$:

$$\omega = x\, y\, z\, dw \wedge dy.$$